MATH11112 REAL ANALYSIS **Week** 03 **lecture** B

Check In

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Reminder:

- a series $C(x) = \sum_{n=0}^{\infty} c_n x^n$ is called a power series in x;
- the set of x such that $C(x)$ is convergent is called the interval of convergence and is of the form $[-R, R]$, $(-R, R]$, $[-R, R)$ or $(-R, R)$; here $R \in [0, +\infty]$ is the radius of convergence;
- convergence is absolute when $x \in (-R, R)$.

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EXAMPLE Determine the radius of convergence R and the interval of convergence of a given power series, and the type of convergence at $x = \pm R$.

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} \ldots$$

We will find R using the Ratio Test:

$$\ell = \lim_{n\to\infty} \frac{|x^{n+1}|/(n+1)}{|x^n|/n} = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| \lim_{n\to\infty} \frac{1}{1+1/n} = |x| \frac{1}{1+0} = |x|$$

If $|x|<1$, then $\ell<1$ so by Ratio Test, the series is absolutely convergent.
If $|x|>1$, $\ell>1$, not abs. convergent. Conclude: $R=1$

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To determine the interval of convergence I, need to test convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ at $x = \pm 1$.

$x = 1$: $\sum_{n=1}^{\infty} x^n/n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ harmonic series

$\Rightarrow$ divergent.

$x = -1$: $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots = (-1) \times$ alternating harmonic series

$\Rightarrow$ CONDITIONAL convergence.

$$I = [-1, 1)$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio Test: $\ell = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} =$

$$= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

abs. convergent FOR all $x \in \mathbb{R}$. $R = \infty$

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Prop (infinite sum of increasing continuous f^{ns})
Suppose each of $f_1(x), f_2(x), \ldots\ldots$ is increasing and continuous on $[c,d]$, and for each $x \in [c,d]$, the series $\sum_{m=1}^{\infty} f_m(x)$ is convergent with sum $F(x)$.
Then: $F(x)$ is continuous on $[c,d]$.

Proof Need to show: $\forall b \in [c,d]$, $\lim_{x \to b} F(x) = F(b)$

$$\Longleftrightarrow \underbrace{\lim_{x \to b-} F(x) = F(b)}_{\text{will prove this}} = \lim_{x \to b+} F(x)$$

won't do - is similar

the $\lim_{n\to\infty} F(x_n)$ for all sequences (x_n) such that (x_n) is strictly increasing and $x_n \to b$ as $n \to \infty$.

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$$a_{m,n} = f_m(x_n) - f_m(x_{n-1}) \qquad n \geq 2,\ m \geq 1$$

Column n:

$$\begin{array}{lll}
f_1(x_2) - f_1(x_1) & f_1(x_3) - f_1(x_2) & f_1(x_4) - f_1(x_3) \cdots \\
f_2(x_2) - f_2(x_1) & f_2(x_3) - f_2(x_2) & f_2(x_4) - f_2(x_3) \cdots \\
f_3(x_2) - f_3(x_1) & f_3(x_3) - f_3(x_2) & f_3(x_4) - f_3(x_3) \cdots \\
\vdots & \vdots & \vdots
\end{array}$$

$$\text{ColumnSum}_n = \sum_{m=1}^{\infty} f_m(x_n) - f_m(x_{n-1})$$

$$= F(x_n) - F(x_{n-1})$$

$$\sum_{n=2}^{\infty} \text{ColumnSum}_n = \sum_{n=2}^{\infty} F(x_n) - F(x_{n-1}) =$$

$$= \big(F(x_2) - F(x_1)\big) + \big(F(x_3) - F(x_2)\big) + \big(F(x_4) - F(x_3)\big) + \cdots$$

$$= \lim_{n\to\infty} \left[F(x_n) - F(x_1) \right] = \left(\lim_{n\to\infty} F(x_n) \right) - F(x_1)$$

$$\text{RowSum}_m = \sum_{n=2}^{\infty} f_m(x_n) - f_m(x_{n-1}) = \Big[\underbrace{\lim_{n\to\infty} f_m(x_n)}_{\substack{\parallel \\ f_m(b)}}\Big] - f_m(x_1)$$

because f_m is cont's and $x_n \to b$ $(n\to\infty)$

$$\sum_{m=1}^{\infty} \text{Row Sum}_m = \sum_{m=1}^{\infty} f_m(b) - f_m(x_1)$$

$$= F(b) - F(x_1).$$

If $a_{m,n}$ are non-negative then the two methods of summation give the same result.

$$a_{m,n} = f_m(x_n) - f_m(x_{n-1}) \geqslant 0$$

$\left\{ \begin{array}{l} x_n > x_{n-1}, \\ f_m \text{ is} \\ \text{increasing so} \end{array} \right.$

$f_m(x_n) \geqslant f_m(x_{n-1})$

Therefore

$$\lim_{n\to\infty} F(x_n) = F(b).$$

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Reminder: a function f is monotone on $[c,d]$ if (f is increasing on $[c,d]$) OR (f is decreasing on $[c,d]$)

Prop

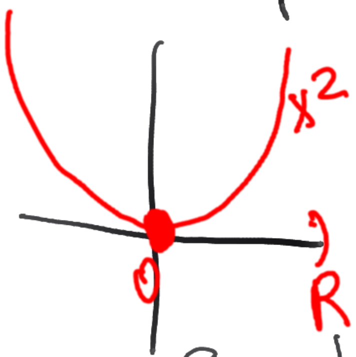
Suppose each of $f_1(x), f_2(x), \ldots\ldots$ is monotone and continuous on $[c,d]$, and for each $x \in [c,d]$, $\sum_{m=1}^{\infty} f_m(x)$ is absolutely convergent and $\sum_{m=1}^{\infty} f_m(x) = F(x)$. Then $F(x)$ is continuous on $[c,d]$.

(Proof in the notes — not in class)
not examinable

THM Let $C(x)$ denote the sum of a power series with radius of convergence R. Then $C(x)$ is a continuous function on $(-R,R)$.

Pf $$C(x) = c_0 + c_1 x + \underbrace{c_2 x^2} + \ldots\ldots$$

$f_m(x) = c_m x^m \leftarrow$ continuous. If m is odd, f_m is monotone everywhere. If m is even, $c_m x^m$ is monotone on $[0,R)$



$\Longrightarrow$ by Prop, $C(x)$ is continuous on $[0,R)$

convergence is absolute

Similarly, $C(x)$ is cts on $(-R,0]$. $\longleftarrow$ it follows that $c(x)$ is cts on $(-R,R)$. ∎