MATH11112 REAL ANALYSIS Week 03 lecture A

Check In ?

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Reminder:

- we started studying convergence of series which aren't non-negative;

- we proved that if a series $\sum a_n$ is absolutely convergent, that is $(*)\ \sum_n |a_n| < +\infty$, then it is convergent, and rearrangements don't change the sum.

Can a series be convergent but not absolutely convergent?

If so, what about rearrangements?

DEF (conditionally convergent)
A series which is convergent, but not absolutely convergent, is conditionally convergent.

[We have: $\sum_{n=1}^{\infty} a_n$: sum exists, but $\sum_{n=1}^{\infty} |a_n| = +\infty$]

THM (Alternating Series Test) Let $a_1, a_2, a_3, \ldots \geq 0$
be a decreasing sequence : $a_1 \geq a_2 \geq a_3 \geq \ldots$
~~THEN the series~~
AND $\lim_{n\to\infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ is convergent.

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Series 1 $\quad (a_1 - a_2) + 0 + (a_3 - a_4) + 0 + (a_5 - a_6) + 0 + \dots$

$+$

Series 2 $\quad a_2 - a_2 + a_4 - a_4 + a_6 - a_6 + \dots$

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

Series 1 has non-negative terms; partial sums:

$$\underline{\underline{\underline{a_1}}} \underbrace{- a_2 + a_3}_{\leq 0} - a_4 \cdots\cdots \underbrace{+ a_{2n-1}}_{\leq 0} \underbrace{- a_{2n}}_{\text{subtract non-negative}} \leq a_1$$

By Boundedness Test, Series 1 is convergent.

Series 2: partial sums $a_2, 0, a_4, 0, a_6, 0, \dots$

$$0 \xleftarrow[n\to\infty]{} 0 \leq (\text{nth partial sum}) \leq a_n \xrightarrow[n\to\infty]{} 0$$

SANDWICH Rule

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so Series 2 is convergent with sum 0.

Both series are convergent, so $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent by A.IS. $\blacksquare$

EX The alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

is convergent: $\left(\frac{1}{n}\right)_{n\geq 1}$ is decreasing with limit 0

(apply the Alternating Series Test)

This series is NOT ABSOL. convergent:

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \left|-\frac{1}{4}\right| + \ldots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = +\infty$$

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Rem $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

Riemann's Rearrangement THM:

if a series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent

then for any real number s there exists a rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ which is convergent with sum s.

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Power series.

DEF Let $c_0, c_1, c_2, \dots$ be real numbers. A series of the form

$$\sum_{n=0}^{\infty} c_n x^n \quad \leftarrow C(x)$$

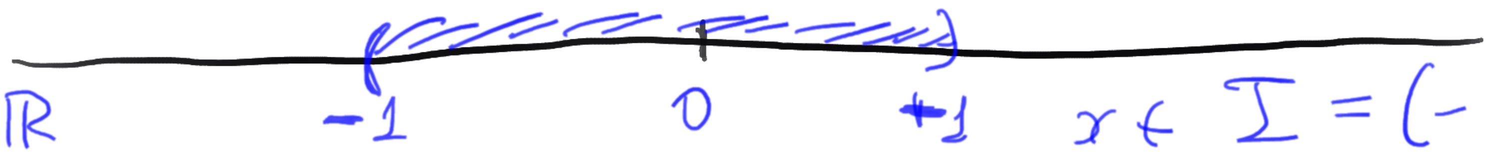
is a **power series** in x.

Ex $G(x) = 1 + x + x^2 + x^3 + \dots$

geometric series, initial term 1, ratio x

$|x| < 1 \Rightarrow G(x)$ is a convergent series

$|x| \geq 1 \Rightarrow G(x)$ is a divergent series



$x \in I = (-1, 1)$

Lemma (absolute convergence for a smaller modulus)

Suppose for a given $x_0 \in \mathbb{R}$, $C(x_0)$ is convergent. Then for all y with $|y| < |x_0|$, $C(y)$ is absolutely convergent.

Proof $\quad C(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ is convergent

$$\Rightarrow \lim_{n\to\infty} c_n x_0^n = 0$$

Nullity Test

$$\Rightarrow \text{(a sequence with a limit is bounded)}$$

$$\exists M > 0 : \quad \forall n \geq 0, \ |c_n x_0^n| = |c_n|\,|x_0|^n \leq M$$

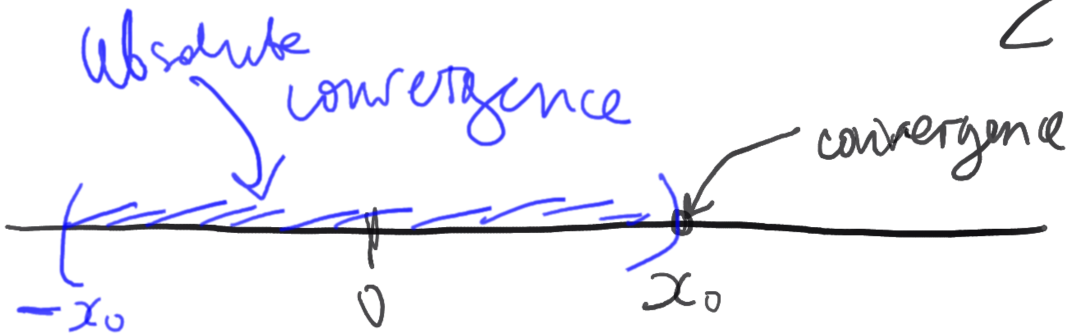
Take $y \in \mathbb{R}$ with $|y| < |x_0|$, set $r = \frac{|y|}{|x_0|} < 1$

$\Rightarrow \sum_{n \geq 0} M r^n$ is convergent (geometric series)

For all $n \geq 0$ we have

$$0 \leq |c_n||y|^n = \overbrace{|c_n x_0^n|}^{\leq M} \cdot \underbrace{\frac{|y|^n}{|x_0|^n}}_{= r^n} \leq M r^n$$

By the Comparison Test, $\sum_{n=0}^{\infty} |c_n y^n| < +\infty$, which means that $\sum_{n=0}^{\infty} c_n y^n$ is absolutely convergent. $\blacksquare$



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Corollary (the set of points of $\mathbb{R}$ where the power series is convergent)

The set of all $t \in \mathbb{R}$ such that $C(t)$ is convergent is one of:

(i) interval $(-R, R)$, $[-R, R]$, $(-R, R]$, $[-R, R)$ where $\underline{R > 0}$;

(ii) $\{0\} \leftarrow R = 0$

(iii) $\mathbb{R} \leftarrow R = \infty$.

And so, there is a unique $R \in [0, \infty]$ such that $C(t)$ is absolutely convergent when $|t| < R$ and is divergent when $|t| > R$.

[Follows from Lemma]

DEF $R \in [0,\infty]$ in Corollary is the radius of convergence $\Bigg\}$ of the power series $C(t)$.
$I = \{t : C(t) \text{ is convergent}\}$ is the interval of convergence

Rem When $x = R$ or $x = -R$, the series $C(x)$ could be
- $*$ absolutely convergent
- $*$ conditionally convergent
- $*$ divergent

Question Given a power series $\sum_{n=0}^{\infty} c_n x^n$, determine:

R = radius of convergence; I = interval of convergence;

$*$ type of convergence when $x = \pm R$.