MATH11112 REAL ANALYSIS

Week 02 Examples class

Check In

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Reminder:

The first written homework is due by Tuesday 11 Feb., 2^{pm}

The exercise sheet and the Gradescope link are in the Week 3 folder on Blackboard.

Plan for today's examples class:

Go through Extra questions on the Week 1 sheet.

MATH11112 Real Analysis (2025). Exercise sheet for week 01

Q2. A "telescoping" series. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(i) Calculate the partial sums s_1, s_2, s_3. Then show that the nth partial sum is $s_n = 1 - \frac{1}{n+1}$.

(ii) Is the series convergent? If so, what is the sum of this series?

Q3. The series of inverse squares and the Basel problem. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(i) By comparing with $\sum \frac{1}{n(n+1)}$ (Q2), show that $s_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ is less than 2 for all n.

(ii) Give a reason why $s_1, s_2, s_3, \ldots$ is an **increasing** sequence.

(iii) State the key result about convergence of a **increasing sequence that is bounded above.**

(iv) Use (i), (ii) and (iii) to conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent with sum $s \leq 2$.

Comment: There is no nice formula for s_n here, yet in 1734 Euler found that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6},$$

Q4. Geometric series and infinite decimals example. You need to know the formulae for the sum $a + ar + \cdots + ar^n$ and the geometric series sum $\sum_{n=0}^{\infty} ar^n$ if $|r| < 1$; review them now.

(i) Let $a_1, a_2, \ldots$ be a sequence of decimal digits, i.e., $a_n \in \{0, 1, \ldots, 9\}$ for all n. The real number represented by the infinite decimal $0.a_1 a_2 a_3 \ldots$ is defined to be the sum of the series

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Use the sum of geometric series to write the real numbers expressed by the infinite decimals $0.5555\ldots$ and $0.9999\ldots$ as rationals p/q where $p, q \in \mathbb{N}$.

(ii) Likewise, calculate $\displaystyle\sum_{n=1}^{\infty} \frac{428571}{10^{6n}}$.

(iii) Show that any series of the form given in (i) is **convergent,** with sum between 0 and 1.

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Extra exercises

Attempt these questions in your own time and compare your answers with the model solutions published on Monday in week 2. Some of these questions will be discussed in the Examples Class on Thursday in week 2.

Q5. The number e is irrational. (i) By showing that partial sums form an increasing sequence and are bounded above by 3, prove that the following series is convergent:

Use the Boundedness Test (this series has non-negative terms!)

$$e = \boxed{1+\frac{1}{1!}+\frac{1}{2!}+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}.}$$

Notation: the sum of the series $\sum_{k=0}^{\infty}\frac{1}{k!}$ is denoted e.

$$S_n = 1+1+\frac{1}{2}+\frac{1}{2\times 3}+\frac{1}{2\times3\times4}+\cdots+\frac{1}{2\times3\times4\times\cdots\times n} \le 1+\left[1+\frac{1}{2}+\frac{1}{2\times2}+\frac{1}{2\times2\times2}+\cdots+\frac{1}{2^{n-1}}\right]$$

$(n! \geq 2^{n-1})$

$$= 1+\left[1\,\frac{1-1/2^n}{1-1/2}\right] \le 1+2=3$$

(ii) Let s_n be the partial sum $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$. Show that $s_n < e < s_n + \frac{1}{n(n!)}$.

The partial sums $s_0, s_1, s_2, \ldots$ are bounded above by 3 so this series with non-neg. terms is convergent with sum ≤ 3

(ii) Prove that $S_n < e \leq S_n + \frac{1}{n(n!)}$ E.g.

if $n=3$, $\frac{1}{3(3!)} = \frac{1}{18}$ so $1+1+\frac{1}{2}+\frac{1}{6} < e \leq 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{18}$,

i.e. $2+\frac{2}{3} < e \leq 2+\frac{13}{18}$

$$2.667 < e \leq 2.722$$

$S_n < e$ obvious, as $e = S_n + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$

$$e - S_n = \underbrace{\frac{1}{(n+1)!}}_{} + \underbrace{\frac{1}{(n+2)!}}_{\frac{1}{(n+1)!\,(n+2)}} + \dots = \sum_{k=n+1}^{\infty} \frac{1}{k!} \leq$$

Comparison test

$$\frac{1}{(n+1)!\,(n+2)(n+3)} \dots$$

$$\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \cdots\right)$$

$$= \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+2}}\right) = \xcancel{\frac{1}{(n+1)!\,\frac{n+1}{n+2}}}$$

$$< \frac{1}{(n+1)!}\,\frac{1}{1-\frac{1}{n+1}} = \frac{1}{(n+1)!\,\frac{n}{n+1}} = \frac{1}{n!\times n}$$

(iii) Prove that e is an irrational number.

Hint: deduce from (ii) that $n!\, s_n < n!\, e < n!\, s_n + 1$ for all n. Note that $n!\, s_n$ is an integer for all $n \in \mathbb{N}$ (*why?*) and so $n!\, s_n$, $n!\, s_n + 1$ are two consecutive integers. Assuming that e is rational, let $n \in \mathbb{N}$ be such that $n!\, e$ is an integer (*why does such an n exist?*) and arrive at a contradiction.

$$s_n < e \leq s_n + \frac{1}{n(n!)} \quad \Big| \times n!$$

$$n!\, s_n < n!\, e < n!\, s_n + \frac{1}{n} \leq n!\, s_n + 1$$

$$n!\, s_n = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$$

$$= \underline{\text{integer}} = A_n$$

$n! \frac{1}{k!}$ is an integer if $k \leq n$.

So $A_n < n!\, e < A_n + 1$ for all $n \in \mathbb{N}$

Assume for contradiction that $e \in \mathbb{Q}$,

then $e = \frac{m}{n}$, $m, n \in \mathbb{N}$.

Then $n!e = (n-1)!\,(\underbrace{ne}_{\in \mathbb{N}}) \in \mathbb{N}$

So we have a natural number

$n!e$ strictly between

two consecutive natural numbers,

$$A_n < n!e < A_{n+1}.$$

This is absurd (a contradiction).

Q6. (A question used in a past exam paper.) If $e^{1/2}$ is sum of the series $\sum_{r=0}^{\infty} \frac{1}{2^r\, r!}$, prove that 671326

$$\boxed{e^{1/2} - \left(1 + \frac{1}{2} + \frac{1}{8}\right) \leq \frac{1}{24}.}$$

$$e^{1/2} = \sum_{r=0}^{\infty} \frac{1}{2^r r!} =$$

$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \ldots$$

$r=1$: $\frac{1}{2^1 \times 1!} = \frac{1}{2}$

$$e^{1/2} - \left(1 + \frac{1}{2} + \frac{1}{8}\right) = \frac{1}{48} + \ldots$$

$$= \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 4!} + \frac{1}{2^5\, 5!} + \ldots$$

$$\leq \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 3!} + \frac{1}{2^5 \cdot 3!}$$

$\overline{\text{geometric series}}$ $\frac{1}{2^3 \times 3!}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) = \frac{1}{48} \times 2 = 1/24$

Q7. The Cauchy condensation test. (i) Try to prove the following theorem (or work through a proof given in the literature):

Suppose $a_1, a_2, a_3 \dots$ are non-negative reals such that $a_1 \geq a_2 \geq a_3 \geq \dots$. The series

(a)
$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

is convergent if, and only if, the "condensed" series

(b)
$$\boxed{a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \sum_{k=0}^{\infty} 2^k a_{2^k}}$$

is convergent.

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(ii) Apply the above theorem (known as the Cauchy condensation test) to show that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergent. (Don't forget: the test is applicable only to non-negative series where $a_1, a_2, a_3 \dots$ is decreasing.)

Q8. Riemann's zeta function. Let s be a positive real number. Use the Cauchy condensation test from Q7 to show that

(i) if $s > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^s} < +\infty$; (ii) if $s \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^s} = +\infty$.

$$\frac{1}{1^s} \geq \frac{1}{2^s} \geq \frac{1}{3^s} \geq \dots$$

$s \in (0, +\infty)$

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\dots$$

Condensed series: $1 \times \frac{1}{1^s} + 2\frac{1}{2^s} + 4\frac{1}{4^s} + 8\frac{1}{8^s} + \dots\dots$

$$= 1^{1-s} + 2^{1-s} + 4^{1-s} + 8^{1-s} \dots\dots = \sum_{k=0}^{\infty} (2^k)^{1-s} = \frac{1}{1-2^{1-s}}$$

if $2^{1-s} < 1$

$1-s<0$

Comment: the zeta function $\zeta(s)$ is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. For example, $\zeta(2) = \frac{\pi^2}{6}$, by a result of Euler. It turns o[illegible]) ca[illegible] be extended to real numbers $s < 1$ (by a different formula) and moreover to al[illegible] $\neq 1$ as a continuous, and differentiable, function of s. The **Riemann Hypoth[illegible]** [illegible]ut about this function $\zeta(s)$ and is one of the most famous unsolved problems in mathematics.