MATH11112 REAL ANALYSIS

Week Ø2 lecture B

Check In ?

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Reminder:

- we are summing double series $(a_{m,n})_{m,n\geq 0}$
- assuming that $a_{m,n}$ are non-negative, we proved that

$S =$ sum by all enumerations = sum by squares = sum by diagonals

- we still need to prove that sum by rows $= S$

(we won't prove for columns as this is similar to rows)

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Square $\text{Sum}_{b\times b}$

$$\begin{array}{ccccccc}
a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

I claim that $S \leq \sum_{m=0}^{\infty} \text{RowSum}_m$.

$a_{00} + a_{01} + \dots + a_{0b} \leq \text{RowSum}_0$

$\vdots$

$a_{b0} + a_{b1} + \dots + a_{bb} \leq \text{RowSum}_b$

$\Downarrow$

$\text{Square Sum}_{b\times b} \leq \sum_{m=0}^{b} \text{RowSum}_m$.

Let $b \to \infty$:

$$\lim_{b\to\infty} \quad \leq \quad \lim_{b\to\infty}$$

$$S \leq \sum_{m=0}^{\infty} \text{RowSum}_m$$

$$\begin{bmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

I now claim that 894863

$$S' \geq \sum_{m=0}^{\infty} \text{RowSum}_m = \sup_M \sum_{m=0}^{M} \text{RowSum}_m$$

It is enough to show that for any fixed M,

$$\underline{\sum_{m=0}^{M} \text{RowSum}_m} \leq S'.$$

$$\text{RowSum}_0 = \lim_{n\to\infty} a_{00} + a_{01} + \dots + a_{0n}$$

$$\text{RowSum}_1 = \lim_{n\to\infty} a_{10} + a_{11} + \dots + a_{1n}$$

$$\vdots$$

$$\text{RowSum}_M = \lim_{n\to\infty} a_{M0} + a_{M1} + \dots + a_{Mn}$$

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$$\mathrm{RowSum}_0 + \mathrm{RowSum}_1 + \cdots + \mathrm{RowSum}_M \overset{(\mathrm{AoL})}{=} \lim_{n\to\infty} \underbrace{\begin{pmatrix} a_{00} + a_{01} + \cdots + a_{0n} \\ + a_{10} + a_{11} \cdots + a_{1n} \\ \cdots \\ + a_{M0} + a_{M1} + \cdots + a_{M,n} \end{pmatrix}}_{\underset{n \geq M}{\leq} \mathrm{SquareSum}_{n\times n}} \leq S$$

Remark In particular, for non-negative double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}$$

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Example

$\text{RowSum}_m = 0 \quad \forall m$

$$
\begin{array}{cccccccc}
1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & -1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & \boxed{0} & \cdots \\
\cdots & & & & & &
\end{array}
$$

MISTAKE!

SHOULD BE -1

$\text{ColumnSum}_0 = 1$

$\text{ColumnSum}_n = 0 \quad \forall n \geq 1$

$$0 = \sum_{m=0}^{\infty} \underbrace{\sum_{n=0}^{\infty} a_{m,n}}_{\text{RowSum}_m} \quad ; \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} = 1 + 0 + 0 + \cdots$$

$$= 1$$

Series with +ve and −ve terms.

THM (Nullity Test for divergence)

If $a_n \not\to 0$ as $n\to\infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof Let $s_n = a_1 + a_2 + \cdots + a_n$

If $\sum_{n=1}^{\infty} a_n$ is convergent, $\exists s \in \mathbb{R}$: $\lim_{n\to\infty} s_n = s$.

(MFA) $\Rightarrow \lim_{n\to\infty} s_{n-1} = s$.

$a_n = s_n - s_{n-1}$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1})$

$\overset{\text{AoL}}{=} \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$. ∎

We proved the contrapositive.

Ex If $a_n \to 0$ as $n \to \infty$, the test does not give us information:

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = +\infty$$

but $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$.

PROP (the Algebra of Infinite Sums, A.o.I.S.)

If $\sum_{n=1}^{\infty} a_n$ is convergent with sum S,

$\sum_{n=1}^{\infty} b_n$ is convergent with sum T,

$\lambda, \mu \in \mathbb{R}$, then

$\sum_{n=1}^{\infty} (\lambda a_n + \mu b_n)$ is convergent with sum $\lambda S + \mu T$.

Pf

$$s_n = a_1 + a_2 + \cdots + a_n$$
$$t_n = b_1 + b_2 + \cdots + b_n$$

Partial sum $(\lambda a_1 + \mu b_1) + (\lambda a_2 + \mu b_2) + \cdots + (\lambda a_n + \mu b_n)$

equals

$$\lambda s_n + \mu t_n$$

$$\lim_{n\to\infty} (\lambda s_n + \mu t_n) \underset{\text{AoL}}{=} \lambda \lim_{n\to\infty} s_n + \mu \lim_{n\to\infty} t_n$$
$$= \lambda S + \mu T. \quad \blacksquare$$

DEF A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

if $\sum_{n=1}^{\infty} |a_n| < +\infty$.

THM Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Then:

1. $\exists\, p_n, q_n$ non-negative: $\sum_{n=1}^{\infty} p_n < +\infty$, $\sum_{n=1}^{\infty} q_n < +\infty$

   and $\quad a_n = p_n - q_n$ (for all n)

2. $\sum_{n=1}^{\infty} a_n$ is convergent [the absolute convergence THM]

3. $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$ [infinite triangle inequality]

PROOF 1. NOTATION: $a \in \mathbb{R}$, $\quad a^+ = \begin{cases} a, & \text{if } a \geq 0 \\ 0, & \text{if } a < 0 \end{cases}$

$$a = a^+ - a^-$$
$$|a| = a^+ + a^-$$

$$a^- = \begin{cases} 0, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Put $p_n = a_n^+$, $q_n = a_n^-$ $(p_n - q_n = a_n)$

$0 \le p_n \le |a_n|$, $0 \le q_n \le |a_n|$

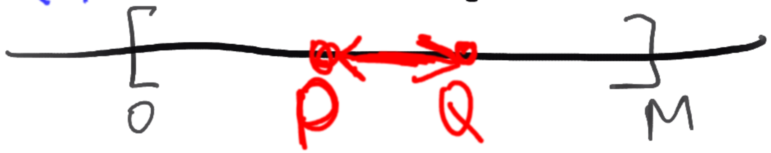
$$\sum_{n=1}^{\infty} |a_n| < +\infty \xRightarrow[\text{Test}]{\text{Comparison}} P = \sum_{n=1}^{\infty} p_n, \quad 0 \le P \le M^*$$

$$Q = \sum_{n=1}^{\infty} q_n, \quad 0 \le Q \le M^*$$

where $M = \sum_{n=1}^{\infty} |a_n|$.

② $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (p_n - q_n) \xRightarrow{\text{AoIS}} \sum_{n=1}^{\infty} a_n$ is convergent with sum $P - Q$.

③ $\left|\sum_{n=1}^{\infty} a_n\right| = |P - Q| \le M$. $\blacksquare$



CLAIM (corollary): (a) All rearrangements of an absolutely convergent series are convergent, with the same sum.

[Using AoIS and this THM]

(b) If $\sum_{m,n} |a_{m,n}| < +\infty$ in a double series,

then this double series satisfies the Rearrangement result we proved for non-negative double series.