MATH11112 REAL ANALYSIS **Week** 02 **lecture** A

Check In

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Last week:

- series (def^n)
- geometric series
- series with non-negative terms
- convergence test

Today:

- Harmonic series
- Rearrangements
- Double series

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SEAtS 665778

PROP The Harmonic Series,
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n} = +\infty$$
(is divergent)

Pf
$$S_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$$
$$S_{2n} = 1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}$$
$$S_{2n}-S_n = \underbrace{\frac{1}{n+1}+\cdots+\frac{1}{2n}}_{\substack{n \text{ numbers} \\ \text{each} \geqslant \frac{1}{2n}}} \geqslant n\times\frac{1}{2n}=\frac{1}{2}$$

$S_1 = 1$, $S_2 - S_1 \geqslant \frac{1}{2} \Rightarrow S_2 \geqslant S_1 + \frac{1}{2}$

$S_4 - S_2 \geqslant \frac{1}{2} \Rightarrow S_4 \geqslant S_1 + 2\times\frac{1}{2}$, and so on:

$S_{2^n} \geqslant S_1 + n\times\frac{1}{2}$ unbounded $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n} = +\infty$. ■

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DEF

Rearrangement: a series $\sum_{n=1}^{\infty} b_n$ is a rearrangement of a series $\sum_{n=1}^{\infty} a_n$ if there exists a bijective function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$ for all $n \in \mathbb{N}$.

THM (Rearrangement of non-neg. series)

If all terms of $a_1 + a_2 + a_3 + \cdots$ are non-negative then all rearrangements of this series have the same sum, and if $a_1 + a_2 + \cdots = +\infty$ then all rearrangements are also divergent to $+\infty$.

Pf Let $\sigma : \mathbb{N} \to \mathbb{N}$ be bijection so that we have $a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + \cdots$

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Partial sums: $a_1 + a_2 + \dots + a_n = S_n$

$a_{\sigma(1)} + a_{\sigma(2)} + \dots + a_{\sigma(n)} = t_n$

if $M = \max(\sigma(1), \sigma(2), \dots, \sigma(n))$, then

$t_n \leq \underline{S_{M(n)}} \leftarrow$ the same or more non-negative terms than t_n.

If $\sum_{n=1}^{\infty} a_n = S < +\infty$ then $t_n \leq S_{M(n)} \leq S$ so the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent with sum $\leq S$ by Boundedness Test. So all rearrangements are convergent with the same or smaller sum.

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Note: $\sum_{n=1}^{\infty} a_{\sigma(n)} \xrightarrow[\sigma^{-1}]{\text{rearrangement}} \sum_{n=1}^{\infty} a_n$

So $\sum_{n=1}^{\infty} a_n$ is a rearrangement of all of its rearrangements!

So $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} a_{\sigma(n)}$

Earlier showed: $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} a_{\sigma(n)}$

So $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$.

Finally, $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent

$\sum_{n=1}^{\infty} a_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_{\sigma(n)}$

By contrapositive

$\sum_{n=1}^{\infty} a_n = +\infty \Rightarrow \sum_{n=1}^{\infty} a_{\sigma(n)} = +\infty$ $\blacksquare$

Double series:

How to sum it?

① By enumeration: put all the $a_{m,n}$ in "one line" (single series), take the sum of that single series.

$$\begin{array}{ccccc}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

② $\text{RowSum}_m = a_{m0} + a_{m1} + a_{m2} + \cdots\cdots$ (if sum exists).

Take $\sum_{m=0}^{\infty} \text{RowSum}_m$ as the sum of the double series.

③ $\text{ColumnSum}_n = a_{0n} + a_{1n} + a_{2n} + \cdots \rightsquigarrow \sum_{n=0}^{\infty} \text{ColumnSum}_n$

④ $\text{DiagSum}_d =$

$= a_{d0} + a_{d-1,1} + \dots + a_{0d}$

$= \sum_{m+n=d} a_{m,n}$

$$\begin{array}{ccccc}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

DiagSum_0

DiagSum_1

DiagSum_2

Can try

$\sum_{d=0}^{\infty} \text{DiagSum}_d$?

(5) $$\text{SquareSum}_{b\times b} = \sum_{\substack{0 \le m \le b \\ 0 \le n \le b}} a_{m,n}$$

$$\boxed{\begin{matrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{matrix}} \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix}$$
$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$\rightsquigarrow$ $\lim_{b\to\infty} \text{SquareSum}_{b\times b}$ should be viewed as the sum of the double series.

<u>PROP</u> If all the $a_{m,n}$ are non-negative, and some enumeration of all the terms gives a single series with sum S, then:

(1) All enumerations result in series with sum S,

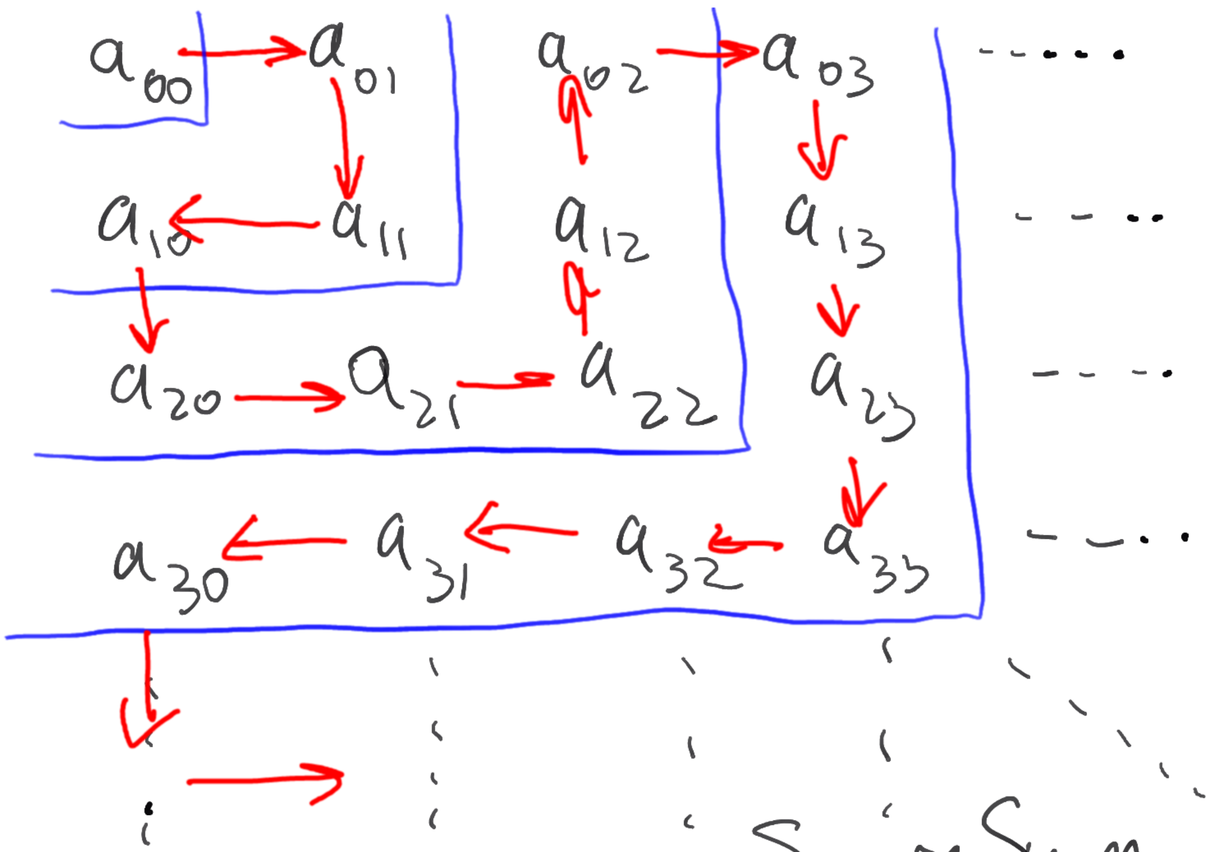
(2) $\lim_{b\to\infty} \text{SquareSum}_{b\times b} = S$,

$$\sum_{d=0}^{\infty} \text{Diag Sum}_d = S,$$

③ $$\sum_{m=0}^{\infty} \text{Row Sum}_m = S, \quad \sum_{n=0}^{\infty} \text{Column Sum}_n = S.$$

Pf ① All different ways to arrange the $a_{m,n}$ into a single series will be rearrangement of one another, so by Rearr. THM for non-neg. series they will have the same sum S.

②



$a_{00} + a_{01} + a_{11} + a_{10} + a_{20} + a_{21} + a_{22} + \dots$

is an arrangement into a series.

It has sum S (part ①)

$\text{SquareSum}_{b\times b}$ form a subsequence of partial sums of this series $\Rightarrow$ $\lim_{b\to\infty} \text{SquareSum}_{b\times b} = S$

$$
\begin{array}{ccccc}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Arranging by diagonals allows us to prove (similarly to squares) that

$$\sum_{d=0}^{\infty} \mathrm{DiagSum}_d = \lim_{n\to\infty}\left(\mathrm{DiagSum}_0 + \mathrm{DiagSum}_1 + \dots + \mathrm{DiagSum}_n\right) = S$$