MATH11112 REAL ANALYSIS Week 01 lecture B

Check In ?

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In the first lecture earlier this week, we proved the Inverse Function Theorem (IFT) for a continuous strictly increasing function on $[a,b]$.

The IFT gives us continuous functions $\sqrt[p]{y}$ $(p \in \mathbb{N})$ and, more generally, y^{α} where $\alpha \in \mathbb{Q}$ (see notes). Later, the IFT will be used to construct $\ln(y)$ and will lead to the Inverse Rule of Differentiation.

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Today: new topic, Infinite series.

DEF A series is an expression of the form $\sum_{n=1}^{\infty} a_n$, where $a_1, a_2, a_3, \dots$ are real numbers.

Alternative notation: $a_1 + a_2 + a_3 + \cdots$, $\sum_{n\geq 1} a_n$, or $\sum a_n$

The n^{th} partial sum of the series is $a_1 + a_2 + \dots + a_n$:
$$s_n = \sum_{i=1}^{n} a_i.$$
If the sequence $s_1, s_2, \dots$ of partial sums has a limit: $\lim_{n\to\infty} s_n = s$, we say that the series $\sum_{n=1}^{\infty} a_n$ is CONVERGENT with sum s.

NOTATION: $\sum_{n=1}^{\infty} a_n = s$

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Rem ① A divergent series is a series which is not convergent.

② Terms of a series can be numbered from 0 or from $N \in \mathbb{Z}$: $\sum_{n=0}^{\infty} a_n$ Partial sum: $S_2 = a_0 + a_1 + a_2$.

PROP $a, r \in \mathbb{R}$ The geometric series with initial term a and ratio r is

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n .$$

This series is convergent if $|r| < 1$, with sum

$$\frac{a}{1-r} .$$

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Proof $\quad S_n = a + ar + \cdots + ar^n = a(1 + r + r^2 + \cdots + r^n)$

$$(1 + r + r^2 + \cdots + r^n)(1 - r) = 1 - \cancel{r} + \cancel{r} - r^2 + \cdots + r^n - r^{n+1}$$
$$= 1 - r^{n+1}$$

Hence $\boxed{1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}}$

$S_n = a\,\frac{1 - r^{n+1}}{1 - r}$, letting $n \to \infty$ $\left(|r| < 1 \overset{MFA}{\Longrightarrow} r^{n+1} \to 0 \right.$ as $\left. n \to \infty \right)$

By the Algebra of Limits

$$\lim_{n \to \infty} S_n = a\,\frac{1 - 0}{1 - r}. \quad \blacksquare$$

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Convergence of series with non-negative terms.

THM (boundedness test for series with non-neg. terms)

$a_1 + a_2 + a_3 + \cdots\cdots$ Assume $\forall n \in \mathbb{N}$, $a_n \geq 0$.

TFAE: ① The partial sums $s_1, s_2, \cdots$ are bounded above.

② The series is convergent.

If ① & ② hold, the sum $\sum_{n=1}^{\infty} a_n$ is the L.U.B. of the partial sums.

$\sup\{s_n : n \geq 1\}$

Pf $\quad s_{n+1} = s_n + \underbrace{a_{n+1}}_{\geq 0} \geq s_n \quad (\forall n)$ so the sequence $s_1, s_2, \cdots\cdots$ is increasing.

MFA: an increasing sequence has a limit $\iff$ it is bounded above. So ① $\iff$ ②.

Moreover, $\lim_{n\to\infty} S_n = \sup\{S_n\}_{n\geq 1}$ for an increasing sequence $(S_n)_{n\geq 1}$. $\blacksquare$

So, if $a_n \geq 0$ $\forall n$, then: $\sum_{n=1}^{\infty} a_n$

convergent with finite sum: $\sum_{n=1}^{\infty} a_n = S$

divergent. notation $\sum_{n=1}^{\infty} a_n = +\infty$

THM (the comparison test for non-negative series)

Assume $0 \le a_n \le b_n \quad \forall n \in \mathbb{N}$.

If $b_1 + b_2 + b_3 + \cdots$ is a convergent series, then (with sum T)

$a_1 + a_2 + \cdots$ is also convergent. (with sum $\le T$)

(If $a_1 + a_2 + \cdots$ is divergent, then $b_1 + b_2 + \cdots$ is also divergent.)

Proof (...) is by contrapositive.

Assume $b_1 + b_2 + \cdots$ is convergent. Write

$$\left.\begin{array}{l} S_n = a_1 + a_2 + \cdots + a_n \\ \;\rotatebox{90}{$\le$}\quad\;\; \rotatebox{90}{$\le$} \qquad\qquad\;\; \rotatebox{90}{$\le$} \\ t_n = b_1 + b_2 + \cdots + b_n \end{array}\right\} \quad S_n \le t_n \le T \quad \forall n$$

$T = \sup\{t_n\}$

So $s_1, s_2, \ldots$ is bounded above by T.
By the Boundedness Test, $\sum_{n=1}^{\infty} a_n$ is convergent,

$$\sum_{n=1}^{\infty} a_n = s \quad \text{where } s = \text{least upper bound of } \{s_1, s_2, \ldots\} \le T.$$

$\blacksquare$

THM (Ratio Test for series with positive terms)
$a_n > 0 \ \forall n \in \mathbb{N}$. Then: suppose that the limit

$$\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ exists. Then}$$

if $\ell < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent
if $\ell > 1$, the series is divergent.

Pf Case $\underbrace{0 \le \ell}_{} < 1$. Choose $\varepsilon > 0$:

ℓ = limit of a positive sequence.

$\ell + \varepsilon < 1$. (For example, $\varepsilon = \frac{1-\ell}{2}$ works).

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \underset{\text{def. of limit}}{\Longrightarrow} \exists N : n \geq N \Rightarrow \frac{a_{n+1}}{a_n} < \underbrace{\ell+\varepsilon}_{= r}$

So $a_{N+1} < r a_N$, $a_{N+2} < r a_{N+1} < r^2 a_N$, ...,

$a_{N+k} < r^k a_N$, for all $k \geq 0$.

$$a_1 + a_2 + \cdots + a_{N+k} \leq (a_1 + a_2 + \cdots + a_{N-1}) + a_N + a_N r + a_N r^2 + \cdots + a_N r^k$$

$$\le (a_1 + \cdots + a_{N-1}) + \frac{a_N}{1-r}, \text{ a finite constant.}$$

Thus, the partial sums of the series $a_1 + a_2 + \cdots$ are bounded above. By the Boundedness Test, the series is convergent.

$\ell > 1$: take $\varepsilon > 0$ such that $\ell - \varepsilon \geq 1$.
($\varepsilon = \ell - 1$ works)

$$\exists N : \forall n \geq N, \quad \frac{a_{n+1}}{a_n} > \ell - \varepsilon \geq 1$$

So $a_N < a_{N+1} < a_{N+2} < \cdots\cdots$

$$S_{N+k} \geq (a_1 + a_2 + \cdots + a_N) + a_N + a_N + \cdots + a_N$$
$$\geq k \cdot a_N \text{ (unbounded)} \Rightarrow \text{divergent.}$$