MATH11112 REAL ANALYSIS   Week ~~0~~ 1   lecture A

Check In

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Lecturer: Yuri Bazlov (w1-6) $\circledast$
Mark Coleman (w6-11)

Lectures: Monday & Wednesday

Examples class: Thursday 3pm
week 2, 4, ~~6~~, 8, 10, 12

Supervision class: Thurs OR Fri
weeks 1, 3, 5, 7, 9, 11
(see your personal timetable)

HOMEWORK: submit via Gradescope
BEFORE 2pm TUESDAY weeks 3, 5, 7, 9, 11

ASSESSMENT: HOMEWORK = 10%
ONLINE QUIZZES = 0%
Week 6 ONLINE TEST = 10%
FINAL EXAM = 80%

MATH11112 REAL ANALYSIS Week 01 lecture A
SEAtS 790599

Def^n (increasing function, strictly increasing, decreasing / strictly decreasing)

$A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ is

increasing: if $\forall x, x' \in A,\ x < x' \Rightarrow f(x) \leq f(x')$

strictly increasing $\Rightarrow f(x) < f(x')$

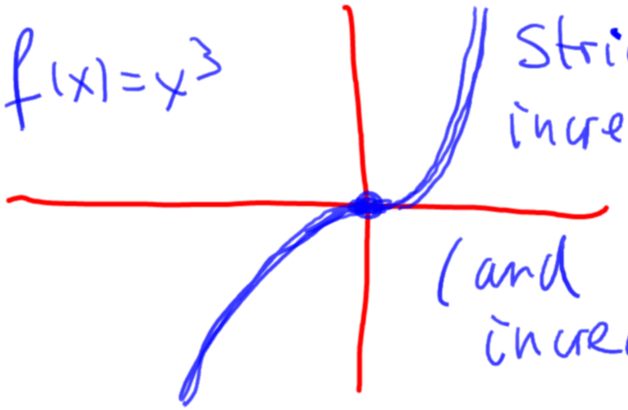
decreasing if $\forall x, x' \in A,\ x < x' \Rightarrow f(x) \geq f(x')$

strictly decreasing $\Rightarrow f(x) > f(x')$

$f(x) = 1$

increasing & decreasing

$f(x) = x^3$



Strictly increasing (and increasing)

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Lemma 1.1 For a function f defined on (a,b) TFAE (the following are equivalent):

1. $$\lim_{x \to b-} f(x) = \ell$$

2. For all strictly increasing sequences (x_n) such that $x_n \to b$ as $n \to \infty$,
$$\lim_{n\to\infty} f(x_n) = \ell.$$

[In words: the limit of f at b from the left is the common limit of all sequences $(f(x_n))_{n\geq 1}$ where a sequence $(x_n)_{n\geq 1}$ is strictly increasing and converges to b.]

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Corollary: $f : (a,b) \to \mathbb{R}$ TFAE:

1. $\lim_{x \to a+} f(x) = \ell$
2. $\forall\ (x_n)_{n \geq 1}$ strictly decreasing, $x_n \to a$ as $n \to \infty$, one has $\lim_{n\to\infty} f(x_n) = \ell$.

THM (the Inverse Function Thm for strictly increasing f's)
A strictly increasing continuous function $f : [a,b] \to [f(a), f(b)]$ has an inverse $g : [f(a), f(b)] \to [a,b]$ which is strictly increasing and continuous.

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Pf $f : [a,b] \to [f(a), f(b)]$ is surjective:
$\forall d \in [f(a), f(b)]$, $\exists\, c \in [a,b] : f(c) = d$.
This is by the Intermediate Value Thm (IVT) for ct^{s} f^{ns} from MFA. (and its corollary)

f is injective: suppose $x_1, x_2 \in [a,b]$, $x_1 \neq x_2$.
either $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$ as f is strictly increasing
or $x_2 < x_1 \longrightarrow f(x_2) < f(x_1)$ so $f(x_1) \neq f(x_2)$
We have shown that f is injective.

So, f is bijective $\underset{\text{MFA}}{\Longrightarrow}$ $\exists$ inverse $g = f^{-1}$:
$[f(a), f(b)] \to [a,b]$

Assume for contradiction that g is <u>not</u> strictly increasing:

$$\neg\left(\forall y_1, y_2 \in [f(a), f(b)],\ y_1 < y_2 \Rightarrow g(y_1) < g(y_2)\right)$$

so

$$\exists\ y_1, y_2 \in [f(a), f(b)] : \underbrace{y_1 < y_2}\ \&\ \underbrace{g(y_1) \geq g(y_2)}$$

f is increasing, so $\quad f(g(y_1)) \geq f(g(y_2))$

f is inverse to g so $\quad \underbrace{y_1 \geq y_2}$ contradiction

$(f \circ g = id_{[f(a), f(b)]})$

which shows that assumption was false, g is strictly increasing.

To prove g continuous: prove that g is c^{ts} at $d \in [f(a), f(b)]$, equivalent to g (MFA)

$$\lim_{y \to d} g(y) = g(d).$$

won't do, it's similar

Equivalently:

$$\lim_{y \to d+} g(y) = \lim_{y \to d-} g(y) = g(d)$$

will focus on this.

Use Lemma 1.1: let (y_n) be strictly increasing sequence with limit d. $y_n \in [f(a), f(b)]$

g is strictly increasing $\Rightarrow$ $(g(y_n))_{n \geq 1}$ strictly increasing sequence in $[a, b]$

$(g(y_n))_{n\geq 1}$ is increasing and bounded above (by b)

$\Rightarrow \exists$ limit : $c = \lim_{n\to\infty} g(y_n)$

f is c^{ts} $\Rightarrow$ $f(c) \underset{\text{MFA}}{=} \lim_{n\to\infty} f(g(y_n)) = \lim_{n\to\infty} y_n = d$

So that $f(c) = d \Rightarrow c = g(d)$. We have shown $\lim_{n\to\infty} g(y_n) = g(d)$ as required. $\blacksquare$

EX $f(x) = x^p$ $(p \in \mathbb{N})$ $f : [0, +\infty) \to [0, +\infty)$

Taking the inverse on $[0, b]$ ($b > 0$ arbitrary) we show, using IFT, $g(y) = \sqrt[p]{y}$ is continuous