Week 4

e^x, $\ln$, differentiation

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Remark: we will use **the Binomial Theorem** which says that for $x, y \in \mathbb{R}$ and $n \geq 0$, $(x+y)^n$ expands as $\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$ where $\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$. The Binomial Theorem is taught in Probability I, and the standard proof is by induction.

The exponential function

Theorem 3.5 allows us to define new continuous functions by power series (with non-zero radius of convergence). Here is the most important example.

Definition: the exponential function and the number e.

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad e = \exp(1).$$

Theorem 4.1: continuity and the law of the exponential.

$\exp$ is a continuous function on $\mathbb{R}$. One has $\exp(x)\exp(y) = \exp(x+y)$ for all x, y.

Proof. Apply the Ratio Test to the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$ to find $\ell = \lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{|x|^n/n!} = \lim_{n\to\infty} |x|/(n+1) = 0$ for all $|x|$. Since $0 < 1$, the power series $\exp(x)$ is absolutely convergent for all $x \in \mathbb{R}$ (the radius of convergence is $R = \infty$). By Theorem 3.5, it follows that $\exp(x)$ is a continuous function on all of $\mathbb{R}$.

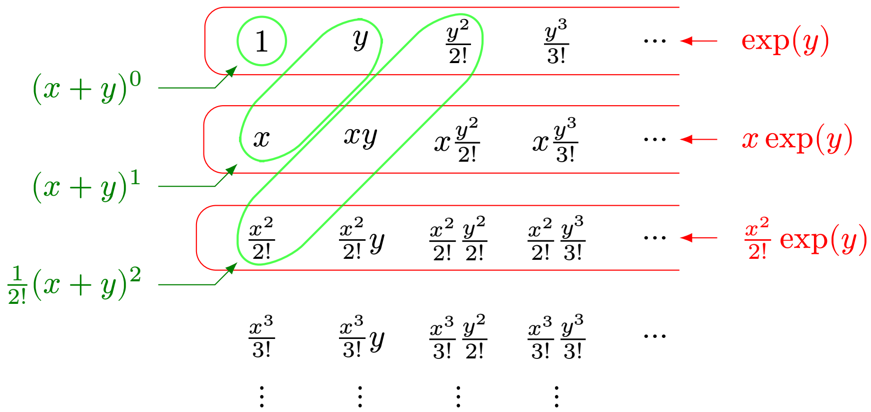

Figure 4.1: *The double series used to prove* $\exp(x)\exp(y) = \exp(x+y)$

To prove $\exp(x)\exp(y) = \exp(x+y)$, we compare two methods of summation of the double series $a_{m,n} = \dfrac{x^m}{m!}\dfrac{y^n}{n!}$, see Figure 4.1. We have

$$\mathrm{RowSum}_m = \frac{x^m}{m!}\left(1 + y + \frac{y^2}{2!} + \dots\right) = \frac{x^m}{m!}\exp(y),$$

hence **summation by rows** gives

$$\sum_{m=0}^{\infty} \mathrm{RowSum}_m = \sum_{m=0}^{\infty} \frac{x^m}{m!}\exp(y) = \exp(x)\exp(y).$$

We now calculate the dth diagonal sum (multiplying and dividing by $d!$ for emphasis):

$$\mathrm{DiagSum}_d = \frac{1}{d!}\left(x^d + \frac{d!}{(d-1)!\,1!}x^{d-1}y^1 + \frac{d!}{(d-2)!\,2!}x^{d-2}y^2 + \dots + y^d\right).$$

By **the Binomial Theorem,** the expression in brackets is the expansion of $(x+y)^d$. Thus, **summation by diagonals** gives

$$\sum_{d=0}^{\infty} \mathrm{DiagSum}_d = \sum_{d=0}^{\infty} \frac{1}{d!}(x+y)^d = \exp(x+y).$$

We claim that the sum of all numbers in this double series does not depend on the method of summation, and so $\exp(x)\exp(y) = \exp(x+y)$. We need to **justify** this claim.

If both x and y are non-negative, then all the numbers $\dfrac{x^m}{m!}\dfrac{y^n}{n!}$ are non-negative. In this case, Proposition 2.3 guarantees that the sum, $\sum_{m,n}\dfrac{x^m}{m!}\dfrac{y^n}{n!}$, is independent of the method of summation, and so $\exp(x)\exp(y) = \exp(x+y)$.

Without the assumption that x, y are non-negative, we can show that the sum of all the absolute values in the table is finite:

$$\left|\frac{x^m}{m!}\frac{y^n}{n!}\right| = \frac{|x|^m}{m!}\frac{|y|^n}{n!} \quad \Rightarrow \quad \sum_{m,n}\left|\frac{x^m}{m!}\frac{y^n}{n!}\right| = \exp(|x|+|y|) < +\infty,$$

so by Claim 2.7, the sum $\sum_{m,n}\dfrac{x^m}{m!}\dfrac{y^n}{n!}$ is still independent of the method of summation, and we still have $\exp(x)\exp(y) = \exp(x+y)$. □

Discussion of the e^x notation. The law of the exponential tells us that, for all $n \in \mathbb{N}$,

$$\exp(n) = \exp(\underbrace{1+1+\dots+1}_{n}) = \exp(1)\exp(1)\dots\exp(1) = e^n.$$

It also follows that, for $p, q \in \mathbb{N}$, $\left(\exp(\frac{p}{q})\right)^q = \exp(q\frac{p}{q}) = \exp(p)$ which is e^p, and so by definition of the qth root and the $(p/q)^{\text{th}}$ power,

$$\exp(\tfrac{p}{q}) = \sqrt[q]{e^p} = e^{\frac{p}{q}}.$$

The law of the exponential also tells us that $\exp(-x)\exp(x) = \exp(0) = 1$, hence

$$\exp(-x) = \frac{1}{\exp(x)} \quad \Rightarrow \quad \exp(-\tfrac{p}{q}) = 1/e^{\frac{p}{q}} = e^{-\frac{p}{q}}.$$

Therefore, $\exp(x) = e^x$ for all rational numbers x. Motivated by this, we extend the notation to all real x:

> **Notation:** e^x.
>
> $\exp(x)$ is written as e^x for all $x \in \mathbb{R}$.

Definition of ln, the natural logarithm function

We are going to introduce the inverse function to e^x. Let us show that e^x is bijective.

> **Proposition 4.2: properties of e^x.**
>
> The function $f(x) = e^x$ is a strictly increasing bijection $\mathbb{R} \to (0, +\infty)$.

Proof. Observe that $x > 0 \implies e^x = 1 + x + \dfrac{x^2}{2} + \cdots > 1 + x$. In particular, e^x is positive for positive x. Then $e^{-x} = 1/e^x$ implies that e^x is positive for all x, and is indeed a function from $\mathbb{R}$ to $(0, +\infty)$.

For all $x, y \in \mathbb{R}$ we have $e^y - e^x = e^x(e^{y-x} - 1)$. If $x < y$, then $e^{y-x} > 1$ as observed above, so $e^y > e^x$. We have shown that e^x is strictly increasing, hence injective.

To show that e^x is surjective, let $d \in (0, +\infty)$ be arbitrary. If $d > 1$, note that $e^d > 1 + d > d$ as shown above. Also $e^0 = 1 < d$. The function is continuous, so by the Intermediate Value Theorem there exists $c \in [0, d]$ such that $e^c = d$.

If $d < 1$ then $\frac{1}{d} > 1$ and by the above, $\frac{1}{d} = e^c$ for some c. We then have $d = e^{-c}$ by the law of the exponential. Finally, if $d = 1$ then $d = e^0$. We have proved that e^x is surjective, and so it is bijective. □

We immediately deduce

> **Theorem 4.3: natural logarithm ln.**
>
> There is a strictly increasing continuous bijection $\ln\colon (0, +\infty) \to \mathbb{R}$ such that $\ln e^x = x$ for all $x \in \mathbb{R}$, $e^{\ln y} = y$ for all $y > 0$ and $\ln(yz) = \ln y + \ln z$ for all $y, z > 0$.

Sketch of proof. e^x is a bijection from $\mathbb{R}$ to $(0, +\infty)$ so it must have an inverse $(0, +\infty) \to \mathbb{R}$, which we denote ln and call the **natural logarithm** function. Inverse means that $\ln e^x = x$ and $e^{\ln y} = y$.

Using the Inverse Function Theorem 1.2, we conclude that $\ln$ is strictly increasing and continuous.

By definition of $\ln$, $x = \ln e^x$ for all x. Set $x = \ln y + \ln z$ to get $\ln y + \ln z = \ln e^{(\ln y + \ln z)}$. By the law of the exponential, this equals $\ln(e^{\ln y} e^{\ln z})$. Yet $e^{\ln y} = y$ and $e^{\ln z} = z$, so the answer simplifies to $\ln(yz)$. We proved the logarithm law, $\ln y + \ln z = \ln(yz)$. $\square$

Differentiation of functions: an informal introduction

We begin the second part of the course: the theory of **differentiation.**

To differentiate a "smooth" function f at point $a \in \mathbb{R}$ means to calculate the **derivative**, $f'(a)$, of f at a. The derivative, if it exists, shows "how fast" the function f grows (or decreases) at the point a. It is impossible to measure growth by looking just at the value of f at a. Rather, the derivative is defined via taking the **limit**; we illustrate this in Fig. 4.2.

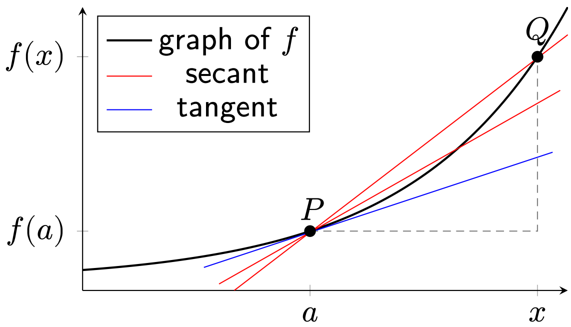

Figure 4.2: The secant passing through the points $(a, f(a))$ and $(x, f(x))$ on the graph is $m = \frac{f(x)-f(a)}{x-a}$. As $x \to a$, we expect the secant to get closer to the tangent at $(a, f(a))$.

We first present the idea **informally** (rigorous definitions are below). Fix a point $P = (a, f(a))$ on the graph of a function f. The slope, or gradient, of the secant passing through P and another point $Q = (x, f(x))$ on the graph is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

As Q "gets closer" to P, the secants "seem" to approach a fixed line, the **tangent** to the graph at P. The gradient of the tangent at P, if it exists, is the derivative of f at a:

$$m_{\text{tangent at } P} = f'(a).$$

Why differentiate functions? It turns out that derivatives appear in powerful results which allow us to approximate functions by extremely good functions — **polynomials** — and to represent some functions as sums of infinite **power series.** But first, we build up theory to

- differentiate basic functions, such as polynomials, rational functions, exponential, logarithm, trigonometric and inverse trigonometric functions;
- use rules of differentiation, to find derivatives of new functions constructed from basic functions.

Definition of the derivative of f at a

We now start our rigorous treatment of differentiation.

> **Definition: open neighbourhood of the point $a \in \mathbb{R}$.**
>
> An **open neighbourhood** of a is an open interval $(a - \delta, a + \delta)$ for some $\delta > 0$.

> **Definition: differentiable at a, derivative at a.**
>
> Let $A \subseteq \mathbb{R}$, and let $f\colon A \to \mathbb{R}$ be a function. **Suppose that $a \in A$ and A contains an open neighbourhood of the point a.** We say that f is **differentiable** at a if
>
> $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
>
> exists. The value of this limit is the **derivative** of f at a, and is denoted $f'(a)$.

Remark: for f to be differentiable at a, $f'(a)$ must be a real number, not infinity.

Definition: differentiable on an open interval.

f is **differentiable on an open interval** I if it is differentiable at every point of I.

Remark: if f is defined on a closed interval $[a, b]$, we will not try to differentiate f at a or at b. Though possible via one-sided limits, we will not need this.

Notation: $\frac{d}{dx}f(x)$.

If a function $f(x)$ is differentiable on an open interval, taking the derivative of f **at each point of the interval** defines a new function. We will write $f'(x)$, or $\frac{d}{dx}f(x)$, to denote the **derivative** of $f(x)$ as a function of x.

There are functions whose derivatives can be computed **by definition,** i.e., by calculating the limit given in the definition of $f'(a)$ without using any further theorems.

Example: derivative of a constant function.

Given $c \in \mathbb{R}$, define a **constant function** on $\mathbb{R}$ by the formula $f(x) = c$ for all x. This function has derivative 0 at all points of $\mathbb{R}$.

Justification: by definition, the derivative at a is $\lim_{x\to a} \frac{c-c}{x-a} = \lim_{x\to a} 0 = 0$.

Remark: Remember that the limit, $\lim_{x\to a} g(x)$, of $g(x)$ as x tends to a, **does not require** $g(x)$ **to be defined at** a**.** Indeed, the MFA definition of limit (revisit it!) looks only at points x such that $0 < |x - a| < \delta$, and this **excludes** the case $x = a$.

For example, the expression $\frac{c-c}{x-a}$ above is **undefined** when $x = a$. But it is of no concern to us: $\frac{c-c}{x-a}$ has value 0 for all x such that $x \neq a$, and so we can write $\lim_{x\to a} \frac{c-c}{x-a} = \lim_{x\to a} 0$.

To conclude: when calculating a limit $\lim_{x\to a}$, we can always assume $x \neq a$.

Example: derivative of the function x.

$\frac{d}{dx}x = 1$ on $\mathbb{R}$.

Justification: by definition, the derivative of x at a is $\lim_{x\to a} \frac{x-a}{x-a} = \lim_{x\to a} 1 = 1$.

Theorem 4.4: differentiable implies continuous.

If f is differentiable at a, then f is continuous at a.

Proof. The criterion of continuity says that f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$. Rearranging, we obtain: f is continuous at $a \iff \lim_{x\to a}(f(x) - f(a)) = 0$.

Assume f is differentiable at a, so that the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$ exists. Then

$$\begin{aligned}\lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) && \text{(can assume } x \neq a\text{)}\\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a}(x-a) && \text{(by AoL for functions)}\\ &= L \cdot 0 = 0.\end{aligned}$$

Thus, f verifies the (rearranged) criterion of continuity above, so is continuous at a. □

Alert: continuous at $a \not\Rightarrow$ differentiable at a.

The converse to Theorem 4.4 does not hold. For example, $f(x) = |x|$ is continuous but not differentiable at 0.

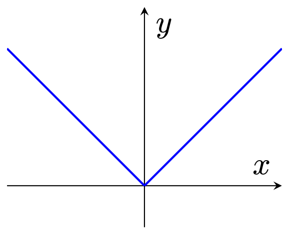

Figure 4.3: *Visibly, the graph of $f(x) = |x|$ is "not smooth" at $x = 0$.*

Justification. "Differentiable at 0" requires the limit $\lim_{x\to 0} \frac{|x|-|0|}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$ to exist. Yet the function is defined by $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}$ see the graph in Fig. 4.3. Hence

$$\lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = 1, \qquad \lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = -1.$$

The one-sided limits are **not** equal, so the limit $\lim_{x\to 0}$ does not exist.

Rules of differentiation: sums and products

We can obtain new differentiable functions from known ones by addition and multiplication.

Theorem 4.5: sum and product rules of differentiation.

Suppose that the functions f, g are differentiable at a. Then

- the function $f+g$ is differentiable at a, and $(f+g)'(a) = f'(a) + g'(a)$;
- the function fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof. **The sum rule (proof not given in class)**: by definition of the function $f+g$, $\dfrac{(f+g)(x) - (f+g)(a)}{x-a}$ is the same as $\dfrac{f(x)+g(x)-(f(a)+g(a))}{x-a}$ which rearranges as $\dfrac{f(x)-f(a)}{x-a} + \dfrac{g(x)-g(a)}{x-a}$. Taking the limit as $x \to a$ and using AoL for functions, we obtain $(f+g)'(a) = f'(a) + g'(a)$ as claimed.

The product rule: by definition, $(fg)(x) = f(x)g(x)$. Start with

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a}$$

where we subtract then add $f(a)g(x)$ in the numerator. The RHS rearranges as

$$\frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}.$$

We are given that g is differentiable at a. Differentiable implies continuous, so g is continuous at a. Hence $\lim_{x\to a} g(x) = g(a)$. Taking $\lim_{x\to a}$ in the last displayed formula and using AoL, we get $f'(a)g(a) + f(a)g'(a)$, as claimed. □

Now, using only $+$ and $\times$, we can construct all **polynomials in** x from constants and the function x. If we apply the rules of differentiation, we obtain

Corollary.

A polynomial in x is differentiable for all $x \in \mathbb{R}$.

Differentiating infinite sums

The sum rule of differentiation **does not** extend to infinite sums. A function defined as a sum of series of differentiable functions may not be differentiable.

Yet one can show that a function defined as a sum of a power series is differentiable on $(-R, R)$, where R is the radius of convergence. We will not go through the proof of this in class. Interested students are invited to construct a proof as an exercise, along the following lines (**not done in class and not examinable**):

Let $f(x) = \sum_{n=0}^{\infty} c_k x^k$ where the radius of convergence is $R > 0$. Let $a \in (-R, R)$. By Algebra of Infinite Sums, we have $f(x) - f(a) = F_a(x)(x-a)$ where $F_a(x) = \sum_{n=1}^{\infty} c_k(x^{k-1} + ax^{k-2} + \cdots + a^{k-2}x + a^{k-1})$. By Proposition 4.6 below, $f(x)$ will be differentiable at a if $F_a(x)$ is shown to be continuous at a.

We note that $F_a(x)$ is obtained if the double series $a_{m,n} = c_{m+n+1}a^m x^n$, $m, n \geq 0$, is summed by diagonals. Yet summation by columns gives the same answer (*this needs to be justified by demonstrating that* $\sum m, n|a_{m,n}| < +\infty$ *when* $a, x \in (-R, R)$) and returns a power series in x. By Theorem 3.5, the sum of a power series is a continuous function, so F_a is continuous on $(-R, R)$, as required.

One conclues from the above that $\left(\sum_{n=0}^{\infty} c_n x^n\right)' = \sum_{n=0}^{\infty} (c_n x^n)' = \sum_{n=1}^{\infty} nc_n x^{n-1}$. So in particular, since $\left(\frac{x^n}{n!}\right)' = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!}$, differentiating the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ term-by-term gives the same series, so $(e^x)' = e^x$.

Instructions for the exam: differentiating a power series term-by-term as above without giving full justification will not be accepted in the exam. If asked to justify differentiation of e^x, give a result obtained below, Proposition 4.7.

Proving “differentiable” by constructing slope function

Rather than showing directly that $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, we may use the following:

Proposition 4.6: differentiability means continuity of the slope function at a.

A function $f(x)$, defined in an open neighbourhood of $a \in \mathbb{R}$, is differentiable at a, if and only if there is a function $F_a(x)$ such that $f(x) - f(a) = F_a(x)(x-a)$ for all x, and $F_a(x)$ is continuous at $x = a$. If these conditions hold, $f'(a)$ equals $F_a(a)$.

Proof. If such F_a exists and is continuous at a, we have $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} F_a(x)$ which, by continuity, is $F_a(a)$. That is, $f'(a)$ exists and equals $F_a(a)$.

Now suppose that f is differentiable at a. Then, defining

$$F_a(x) = \begin{cases} \frac{f(x)-f(a)}{x-a}, & x \neq a, \\ f'(a), & x = a. \end{cases}$$

guarantees $\lim_{x\to a} F_a(x) = F_a(a)$, so by criterion of continuity F_a is continuous at a. □

We call F_a the **slope function** for f at a, because $F_a(x)$ is the slope (the gradient) of the secant through the points $(a, f(a))$ and $(x, f(x))$ on the graph of f. It is useful to note the slope function for the polynomial x^n:

$$f(x) = x^n \quad \Rightarrow \quad F_a(x) = \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \cdots + a^n.$$

This formula defines a polynomial function of x which is continuous everywhere, including at $x = a$. One has $F_a(a) = na^{n-1}$ which is the derivative of x^n at $x = a$.

Differentiating e^x

We use the method of continuous slope function to differentiate e^x.

Proposition 4.7: derivative of e^x.

$\frac{d}{dx}e^x = e^x$.

Proof. To differentiate e^x at 0, write

$$e^x - e^0 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \underset{\text{AoIS}}{=} x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = (x-0)F_0(x).$$

The slope function $F_0(x)$ is the sum of a power series convergent for all x, hence is continuous by Theorem 3.5, and by Proposition 4.6 $\frac{d}{dx}(e^x)|_{x=0}$ exists and equals $F_0(0) = 1$. This proves the **Special Limit for e^x:**

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Indeed, the left-hand side is exactly the derivative of e^x at $x = 0$ which we have just found to be 1. We now differentiate e^x at an arbitrary $x \in \mathbb{R}$:

$$\frac{d}{dx}e^x = \lim_{y\to x} \frac{e^y - e^x}{y - x} = \lim_{y\to x} e^x \frac{e^{y-x} - 1}{y - x} \underset{h=y-x}{=} e^x \lim_{h\to 0} \frac{e^h - 1}{h}.$$

By the Special Limit, this is $e^x \times 1 = e^x$. □

The Chain Rule and the Quotient Rule

We will work in the situation

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

We will write g as a function of $y \in \mathbb{R}$ and f a function of $x \in \mathbb{R}$.

Theorem 4.8: The Chain Rule.

If $g(y)$ is differentiable at $y = k$ and $f(x)$ is differentiable at $x = g(k)$ then $(f \circ g)(y)$ is differentiable at $y = k$, and $(f \circ g)'(k) = f'(g(k))g'(k)$.

Proof. By Proposition 4.6, whenever f is differentiable at a point ℓ, one has

$$f(x) - f(\ell) = F_\ell(x)(x - \ell)$$

for all x, where the slope function F_ℓ is continuous at ℓ. In particular, this holds for $x = g(y)$ and $\ell = g(k)$:

$$f(g(y)) - f(g(k)) = F_\ell(g(y))(g(y) - g(k)) = F_\ell(g(y))G_k(y)(y - k),$$

where we assumed that g was differentiable at k and applied Proposition 4.6 to g.

The function $F_\ell(g(y))$ is continuous at $y = k$, because $g(y)$ is continuous (even differentiable!) at k, F_ℓ is continuous at $g(k) = \ell$, and a composition of continuous functions is continuous. The function $G_k(y)$ is continuous at k. Therefore, by Algebra of Continuous Functions, $F_\ell(g(y))G_k(y)$ is a continuous function of y. It immediately follows by Proposition 4.6 that the function $f(g(y))$ is differentiable at $y = k$, with

$$F_\ell(g(k))G_k(k) = f'(g(k))g'(k)$$

as its derivative at k, as claimed. □

Example.

Find $\frac{d}{dy}e^{-\frac{y^2}{2}}$.

Solution. Put $f(x) = e^x$ and $g(y) = -\frac{1}{2}y^2$ so that our required function is $f(g(y))$. To apply the Chain Rule, we must check that the assumptions of Theorem 4.8 are met:

- $g(y) = -\frac{1}{2}y^2$ is a polynomial, hence is differentiable for all y, with $g'(y) = -y$;
- $f(x) = e^x$ is differentiable for all x by Proposition 4.7, with $f'(x) = e^x$.

Hence we are allowed to use the Chain Rule: $\frac{d}{dy}e^{-\frac{y^2}{2}} = f'(g(y))g'(y) = e^{-\frac{y^2}{2}} \cdot (-y) = -ye^{-\frac{y^2}{2}}$.

Corollary: the Quotient Rule.

If $g(a) \neq 0$ and $f(y)$, $g(y)$ are differentiable at $y = a$, then

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}, \quad \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof. If $h(x) = \frac{1}{x}$ then, for any $\ell \neq 0$, $h'(\ell) = \lim_{x\to\ell} \frac{\frac{1}{x}-\frac{1}{\ell}}{x-\ell} = \lim_{x\to\ell} \frac{\ell-x}{(x-\ell)x\ell}$. When calculating $\lim_{x\to\ell}$, we may assume that $x \neq \ell$, so this simplifies to $\lim_{x\to\ell} \frac{-1}{x\ell} = -\frac{1}{\ell^2}$.

Writing $\frac{1}{g(y)}$ as $h(g(y))$ and applying the Chain Rule, we have $(\frac{1}{g})'(a) = h'(g(a))g'(a) = -\frac{1}{g(a)^2}g'(a)$ as claimed. Now, to obtain $(\frac{f}{g})'$, apply the Product Rule to $f \cdot \frac{1}{g}$. □