§2.2 Trajectories, phase portrait.

Start with 1 D (scalar equation)

\[ \dot{x} = f(x). \]  
(separable ODE)

\[ \int_{x_0}^{x(t)} \frac{dx}{f(x)} = \int_0^t dt \]

\[ \implies t = \int_{x_0}^{x(t)} \frac{dx}{f(x)} = G(x(t)) - G(x_0), \]

solution as an implicit form.

The qualitative behaviour is easier to understand by plotting the "phase portrait".

If \( x(0) = x_0 \in (x_1, x_2) \)

then \( x(t) \to x_2 \)

ODEs \( \leftrightarrow \) a vector field

phase portrait: the union of trajectories for the solutions.

Example 2.6

\[ \dot{x} = -y, \quad \dot{y} = x \]

\[ \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{x}{y} \]

\[ y \, dy + x \, dx = 0 \]

or \[ \frac{d}{dx} \left( \frac{x^2 + y^2}{2} \right) = 0 \]

governing equation for the trajectories.
Example 2.7 (Newtonian Mechanics)

\[ m \ddot{x} = F(x) = -U'(x) \]

\[
\uparrow \\
\text{force} \quad U: \text{the potential}
\]

Introduce \( p = m \dot{x} \) the linear momentum, then

\[
\begin{cases}
\dot{x} = p/m \\
\dot{p} = m \ddot{x} = -U'(x)
\end{cases}
\]

The trajectories are governed by

\[
\frac{dp}{dx} = \frac{\dot{p}}{\dot{x}} = -\frac{U'(x)}{p/m}
\]

or

\[
\frac{p}{m} \frac{dp}{dx} + U'(x) dx = 0
\]

Integrate both sides to get

\[
\frac{p^2}{2m} + U(x) = \text{constant} \quad (\text{we call it the total energy})
\]
The trajectories governed by
\[ \frac{dy}{dx} = \frac{-x + x^3}{y} \]
or
\[ \frac{1}{2} y^2 + \frac{x^2}{2} - \frac{1}{4} x^4 = \text{constant}. \]

Semi-group property of the solution.
\[ x(t) = f(x) \quad x \in \mathbb{R} \]
\[ \text{time independent or autonomous}, \]

if we write the solution as
\[ x(t) = \Psi_t(x_0), \quad x(0) = x_0 \]
then \[ \Psi_0(x_0) = x_0 \]
and \[ \Psi_t(\Psi_s(x_0)) = \Psi_{t+s}(x_0) \]
evolve the solution (starting from \( x_0 \))
first by time \( s \), then
by time \( t \).

For example, a
\[ \Psi_t(x_0, y_0) = (x_0 e^{-t} + te^{-ty_0}, e^{-ty_0}) \]
is a solution to the system
\[ x' = -x + y, \quad y' = -y \]
\[ \Psi_{t+s}(x_0, y_0) = (x_0 e^{-(t+s)} + (t+s)e^{-(t+s)y_0}, e^{-(t+s)y_0}) \]
\[ \Psi_t(\Psi_s(x_0, y_0)) = \Psi_t(x_0 e^{-s} + se^{-s}y_0, e^{-s}y_0) \]
\[ = (x_0 e^{-s} + se^{-s}y_0) e^{-t} + te^{-t}(e^{-s}y_0), \]
\[ = e^{-t} (e^{-s}y_0) \]
\[ = (x_0 e^{-s} + (t+s)e^{-(t+s)y_0}, e^{-(t+s)y_0}) \]
\[ = \Psi_{t+s}(x_0, y_0). \]
(\psi_t(x)) satisfies the semi-group property.

\[ \psi_t(x) \]

\( \psi_t(x) \) is the solution to an autonomous ODE, \( \dot{x} = f(x) \) for some \( f \),

\[ f(x) = \left. \frac{\partial}{\partial t} \psi_t(x) \right|_{t=0} \].

Determine the law of dynamics at initial time.

Example 2.9.

\[ \psi_t(x) = \frac{x_0}{1-tx_0} \]

and the ODE

\[ \dot{x} = x^2 \]

\[ \frac{d}{dt} \psi_t(x) = \frac{x_0^2}{(1-tx_0)^2} = \left( \frac{x_0}{1-tx_0} \right)^2 = \psi_t(x)^2. \]

\[ \left. \frac{d}{dt} \psi_t(x) \right|_{t=0} = x_0^2 \]