Example 2.16 (Predator-prey dynamics)
\[ \begin{align*}
\dot{x} &= (A - BY)x \\
\dot{y} &= (CX - D)y \\
A y &- B y C - C x + D \log x \\
A \log y - B y - C x + D \log x &- \text{ constant.}
\end{align*} \]

\[ S = \{ (x, y) \mid y = 0 \} \]
\[ \begin{align*}
G(x, y) &= y \\
\frac{d}{dt} G(x, y) \bigg|_{G=0} &= y \bigg|_{G=0} \\
&= (CX - D)y \bigg|_{y=0} \\
&= 0
\end{align*} \]

Similarly, for the y-axis (which is invariant).

Example 2.17. We can show that \( y = 2x \)

It is invariant for the system.
\[ \begin{align*}
\dot{x} &= \frac{5}{2} x - \frac{y}{2} + 2x^2 + \frac{1}{2}y^2 \\
\dot{y} &= -x + 2y + 4xy
\end{align*} \]

\[ S = \{ (x, y) \mid y = 2x = 0 \} \] , \[ G(x, y) = y - 2x \]

\( S \) is invariant if and only if

\[ d \frac{G(x, y)}{dt} \bigg|_{G=0} = 0 \]

\[ = y - 2x = -x + 2y + 4xy \]

\[ \bigg|_{y=0} = (5x - y + 4x^2 - y^2) \bigg|_{y=0} \\
&= -x + 4x + 8x \\
&= -(5x - 2x + 4x^2 - 4x^2) \\
&= 0
\]

Example 2.17 (Conserved quantities as invariant set)
\[ r = \text{P/m} \]
\[ \rho = -\nabla U \]
\[ x \in \mathbb{R}^n \]
\[ E(x, p) = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) \]

is conserved, and the set
\[ \{ (x, p) \mid E(x, p) = E(x_0, p_0) \} \]

is invariant.

**Example 2.19 (Invariant sets defined by inequalities):**

we can show that the unit disk \( x^2 + y^2 < 1 \)

is invariant for the system
\[ \dot{x} = -x + y, \quad \dot{y} = -x - y. \]

we only have to check that the points do not escape from the boundary \( x^2 + y^2 = 1 \):

if \( (x, y) \) on the boundary (that is \( x^2 + y^2 = 1 \)),
then
\[ \frac{d}{dt} (x^2 + y^2) = 2x\dot{x} + 2y\dot{y} =
2x(-x+y) + 2y(-x-y) = -2(x^2 + y^2) \leq 0 \]

**Example 2.20 (More general invariant set defined by "boundary" functions):**

\[ X = f(x) \], \quad x \in \mathbb{R}^n \]

the set \( S = \{ x \in \mathbb{R}^n : V(x) < C \} \)
is invariant if and only if
\[ \frac{d}{dt} V(x) \bigg|_{V(x) = C} = x \cdot \nabla V(x) \bigg|_{V(x) = C} = f(x) \cdot \nabla V(x) \bigg|_{V(x) = C} < 0 \]
solutions to ODEs: \( \dot{x} = f(x) \)

The solution may not be unique if \( f \) is not smooth enough, as in the following example.

Example 2.21

\[ \dot{x} = \sqrt{|x|} \quad x(0) = 0 \quad f(x) = \sqrt{|x|} \]

There are infinitely many solutions.

\[ x(t) = \begin{cases} 0 & \text{if } 0 < t < 2 \\ \frac{1}{4} (t-2)^2 & \text{if } t \geq 2 \end{cases} \]

The solution may not exist, if \( f(x) \) increases too fast.

Example 2.22: \( \dot{x} = x^2 \), \( x(0) = x_0 > 0 \)

The ODE is separable, and can be written as

\[ 0 = \frac{dx}{dt} = dt \quad \text{or} \quad -\frac{1}{x(t)} + \frac{1}{x_0} = t \quad \text{or} \quad x(t) = \frac{x_0}{1 - t x_0} \]

As \( t \) approaches \( \frac{1}{x_0} \), \( x(t) \) goes to infinite.
time, we introduce the equivalent integral equation
\[ x(t) = x_0 + \int_0^t f(x(s)) \, ds \]

\[ \frac{dx}{dt} = f(x), \quad x(0) = x_0. \]

The local existence of solutions can be established by the Picard's iteration
\[ x^{(n+1)}(t) = x_0 + \int_0^t f(x^{(n)}(s)) \, ds, \quad n = 0, 1, 2, \ldots \]
\[ x^{(0)}(t) \equiv x_0. \]

Example 2.31. Consider the Picard's iteration for the ODE
\[ \frac{dx}{dt} = ax, \quad x(0) = 1 \]

we know the exact solution is
\[ x(t) = \exp(at). \]

The Picard's iteration is defined as
\[ x^{(n+1)}(t) = 1 + \int_0^t a x^{(n)}(s) \, ds \]
\[ = 1 + a \int_0^t x^{(n)}(s) \, ds. \]

\[ x^{(0)}(t) \equiv 1 \]
\[ x^{(1)}(t) = 1 + a \int_0^t x^{(0)}(s) \, ds = 1 + at. \]
\[
= 1 + a \int_0^t (1 + as) \, ds \\
= 1 + a \left( t + \frac{at^2}{2} \right) \\
= 1 + at + \frac{1}{2} at^2.
\]

Similarly, \( x^{(n)}(t) = 1 + at + \frac{1}{2!} a^2 t^2 + \ldots \)

\[+ \frac{1}{n!} \alpha^n t^n. \]

In general, \( x^{(n)}(t) \) converges to the exact solution, if \( \alpha \) is small enough.