5 Centre manifold and bifurcation

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n. \]

- If \( x^* \) is a fixed point, then we have the linearised system around \( x^* \):
  \[ \frac{d}{dt}(x-x^*) \sim A(x-x^*) \quad A = Df(x^*), \]
  for all \( i \).

- If \( \text{Re} \lambda_i(A) \neq 0 \), the behavior of the linearised system is similar to that of the full system; otherwise, \( \text{if Re} \lambda_i(A) = 0 \) for some \( i \), then the behavior could be different.

- Hyperbolic fixed point.

Bifurcation (qualitative change of behavior on parameters) happens at \( \text{Re} \lambda_i(A) = 0 \).

Simplest example:

1. \[ \dot{x} = \mu x, \]
   \[ \mu > 0, \quad |x| \to \infty \quad (x \to 0). \]
   \[ \mu < 0, \quad |x| \to 0 \]

   \[ \lambda = \mu \]

8.5.1 Centre manifold.

Recall the example: \[ \dot{x} = -x \]
\[ \dot{y} = y + x^2. \]

Linearised system:
\[ \dot{x} = -x, \quad \dot{y} = y \]

\[ W^s \quad W^u \]

Diagram:

\[ W^s \quad W^u \]
\(E^S/E^U\) is the "linearised space" of \(W^S/W^U\) near the origin, in the sense that \(E^S/E^U\) passess the same same fixed point and is tangent to \(W^S/W^U\). If we can find the solution explicitly, \(W^S/W^U\) can be obtained (using that solutions converge to the fixed point as \(t \to \infty\) on \(W^S\)). Alternatively, we can parameterise the stable manifold as

\[
W^S = \{ (x, y) \mid y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \}
\]

determined by the invariant system,

\[
\frac{dy}{dx} = \text{determined by the linearised system,}
\]

because \(E^S = \{ (x, y) \mid y = a_0 + a_1 x \}\).

Since \(E^S = \{ (x, y) \mid y = 0 \}\), we get \(a_0 = 0\): \(E^S/W^S\) passes the origin.

\(a_1 = 0\): given by the eigenvector associated with the eigenvalue \(-1\).

Since \(W^S = \{ (x, y) \mid y = a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \}\) is invariant,

\[
0 = \frac{dy}{dx} = \frac{y - (a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)}{x}
\]

\[
y = (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots) x
\]

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y = (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots) x
\]

\[
y = a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + x^2 +
\]

\[
2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \ldots
\]

\[
(3a_2 + 1)x^2 + 4a_3 x^3 + 5a_4 x^4 + \ldots
\]

\[
a_2 = -\frac{1}{3}, \quad a_3 = a_4 = \ldots = a_n = 0
\]
Theorem (Centre manifold).
If the system $\dot{x} = f(x)$ has a fixed point $x^*$ (assumed to be at the origin) and the eigenvalues can be grouped into three subsets
\[ \sigma_+ = \{ \lambda_i(A) : \text{Re} \lambda_i(A) > 0 \} \]
\[ \sigma_0 = \{ \lambda_i(A) : \text{Re} \lambda_i(A) = 0 \} \]
\[ \sigma_- = \{ \lambda_i(A) : \text{Re} \lambda_i(A) < 0 \} \]

there exists a transformation from
\[ x \in \mathbb{R}^n \rightarrow (\tilde{x}^+, \tilde{x}^-) \in \mathbb{R}^{2n}. \]

\[ \tilde{x}^+ = A^+ x^+ \]
\[ \tilde{x}^- = A^- x^- \]
\[ x^c = g(x^c) \]

There are also unstable/central/stable manifolds
\[ W^u, W^c, W^s. \]

[Diagram showing manifolds and tangent spaces]
Example 5.2: \[ \begin{align*} 
& x = xy \\
& y = -y - x^2 
\end{align*} \]

Linearised system:
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Two eigenvalues:
\[ \lambda_1 = 0, \quad \lambda_2 = -1 \]

Eigenvectors:
\[ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ E_c \text{ the linear space spanned by eigenvectors associated with } 0 \text{ eigenvalue,} \]
\[ E_c = \{ (x, 0) \mid x \in \mathbb{R} \} \]

\[ W_c = \{ (x, y) \mid y = a_2 x^2 + a_3 x^3 + \ldots \} \]

Since \( W_c \) is invariant,
\[ \begin{align*} 
0 &= \frac{d}{dt} \left( y - (a_2 x^2 + a_3 x^3 + \ldots) \right) \\
&= \left( y - (2 a_2 x + 3 a_3 x^2 + \ldots) \right) x y \\
&= (-y - x^2 - (2 a_2 x + 3 a_3 x^2 + \ldots) x y) / (a_2 x^2 + a_3 x^3 + \ldots) \\
&= -(a_2 x^2 + a_3 x^3 + \ldots) - x^2 - (2 a_2 x + 3 a_3 x^2 + \ldots) x (a_2 x^2 + a_3 x^3 + \ldots) \\
&= -(a_2 + 1) x^2 - a_3 x^3 - (a_4 + 2 a_2^2) x^4 + \ldots \\
\end{align*} \]

\[ a_2 = -1, \quad a_3 = 0, \quad a_4 = -2 a_2^2 = -2 \]

General procedure to find \( W_c \):
1. Find the fixed point \( x^* \), Jacobian matrix \( A = Df(x^*) \), eigenvalue, eigenvector associated with \( \lambda = 0 \).
2. Find \( E_c \), and the representation of \( W_c \).
3. Find the higher order coefficients, using \( W_c \) is invariant.