3.5 Linearisation and non-linear terms.

(1) When linearised system gives a 'good' description of the original non-linear system near the fixed point:
- When $\text{Re} \lambda_i \neq 0$ for all $\lambda_i$.
(2) The general structure of solutions:
- Stable/unstable/centre manifold.

**Example 3.10** Consider the system:

\[
\begin{align*}
\dot{x} &= -x, \\
y &= y + x^2.
\end{align*}
\]

With the linearised system (near origin):

\[
\begin{align*}
\dot{x} &= -x, \\
y &= y.
\end{align*}
\]

From $\dot{x} = -x$, we get $x(t) = x_0 e^{-t}$, then

\[
\begin{align*}
y &= y + x^2 = y + x_0^2 e^{-2t} \\
\frac{d}{dt}(e^t y) &= e^t (y - y) = x_0^2 e^{-3t}.
\end{align*}
\]
Integrating on both sides,

\[ e^{-t} y = y_0 + x_0 \frac{1 - e^{-3t}}{3} \quad \text{or} \]

\[ y = y_0 e^t + x_0 \frac{e^{t} - e^{-2t}}{3} \]

Question: when \((x(t), y(t)) \rightarrow (0, 0)\) as \(t \rightarrow \infty\)?

Answer: \(y_0 + \frac{2}{3} = 0\), so that \(y(t) = -\frac{1}{3} x_0 e^{-2t} \rightarrow 0\) as \(t \rightarrow \infty\)

\(E^S = \int x \in \mathbb{R}^n, \text{ so that the solution } y(t)\)

for the linearized system converges to the origin, as \(t\) goes to \(\infty\).

\(W^S = \int x \in \mathbb{R}^n, \text{ so that the solution } y(t)\)

of the full system converges to the corresponding fixed point as \(t\) goes to \(\infty\).

Similarly for \(E^u\) and \(W^u\), by replacing \(t \rightarrow +\infty\) by \(t \rightarrow -\infty\).

Theorem (relation between the full nonlinear and its linearized system). Consider

\[ \dot{x} = Ax + O(1x^2) \quad \text{or higher order terms} \]

If \(A\) has no eigenvalue with zero real part, then there exist a coordinate transformation \(x \rightarrow y = (y_1, y_2) = Ux\)

\[ \dot{y}_1 = A^+ y_1 + O(1|y|^2) \]

\[ \dot{y}_2 = A^- y_2 + O(1|y|^2) \]
so that \( \lambda_i(A^+) \) are the eigenvalues of \( A \) with positive real part, and \( \lambda_i(A^-) \) are the eigenvalues of \( A \) with negative real part.

Moreover, there are two invariant manifolds \( W^s \) and \( W^u \), called stable and unstable manifolds, and \( W^s \) is tangent to \( E^s \)
\( W^u \) is tangent to \( E^u \) (the stable/unstable manifold of the linearised system).

Theorem (Hartman–Grobman) The trajectories of the full non-linear system \( \dot{x} = f(x) \)
for a fixed \( x^* \) (\( f(x^*) = 0 \)) is “similar” to the linearised system
\[
\dot{x} = A(x-x^*) \quad A = Df(x^*),
\]
provided that \( \text{Re} \lambda_i(A) \neq 0 \) for all \( i \).

Example 3.11 \( \dot{x} = y \)
\( \dot{y} = x^2 + x \).

The trajectories governed by the ODE
\[
\frac{dy}{dx} = \frac{x^2 + x}{y}
\]
which is separable.

Therefore, the trajectories are given by
\[
\frac{1}{2} y^2 = \frac{1}{3} x^3 + \frac{1}{2} x^2 + C,
\]
\( W^u \Rightarrow f(x,y) / \frac{1}{2} y^2 = \frac{1}{3} x^3 + \frac{1}{2} x^2 \).
Example 3.12 \[ \dot{x} = x \]
\[ \dot{y} = y^2 \]

linearised system

\[ \begin{align*}
\dot{x} &= x \\
\dot{y} &= 0
\end{align*} \]