§ 6 Bifurcation of maps

§ 6.1 Fixed points, periodic orbits, and their stability.

\[ x_{n+1} = f(x_n) \]

fixed points: \[ x^* = f(x^*) \]

linear stability:

\[ x_{n+1} - x^* = f(x_n) - f(x^*) \]

\[ \approx f'(x^*)(x_n - x^*) \]

if \( x_0 \) is close to \( x^* \),

\[ x_{n+1} \text{ is close to } x^* \] (or \( x_n \to x^* \)),

if \( |f'(x^*)| < 1 \).

periodic orbits of order \( p \)

\( (x_0, x_1, \ldots, x_{p-1}) \) such that

\[ x_{n+p} = f(f \cdots (f(x_n))) = f^p(x_n) \]

The stability of this orbit is equivalent to the stability of any \( x_0, 0 \leq k \leq p-1 \), for the map

\[ z_{n+1} = F(z_n), \quad F(z) = f^p(z) \]

Some facts:

1. \( x_0, 0 \leq k \leq p-1 \), is a fixed point of \( z_{n+1} = F(z_n) = f^p(z_n) \), or

\[ x_0 = f^p(x_0) \]

2. \( \frac{d}{dx} F(z) \bigg|_{x_0} = \frac{d}{dx} F(z) \bigg|_{x_1} = \cdots = \frac{d}{dx} F(z) \bigg|_{x_{p-1}} \]

\[ = f'(x_0) f'(x_1) \cdots f'(x_{p-1}) \]

For example, if \((x_0, x_1)\) is a periodic orbit of order two, then
\[ x_1 = f(x_0), \quad x_0 = f(x_1) \]

then \[ x_1 = f(x_0) = f(f(x_1)) \] and \[ x_0 = f(x_1) = f(f(x_0)) \]

\[
\frac{d}{dx} f(f(x)) \bigg|_{x_0} = f'(f(x_1)) f'(x_1) \bigg|_{x_0} = f'(f(x_0)) f'(x_0) = f'(x_1) f'(x_0),
\]

Similarly, \[ \frac{d}{dx} f(f(x)) \bigg|_{x_1} = f'(x_1) f'(x_0) \].

Hence, a periodic orbit \( x_0, x_1, \ldots, x_{p+1} \) is linearly stable if

\[
\left| \frac{1}{f'(x_0) f'(x_1) \cdots f'(x_{p+1})} \right| < 1
\]

**Invariant Set**: S is an invariant set for the map \( x_{n+1} = f(x_n) \), if \( x_0 \in S \), then \( x_n \in S \) for all \( n \geq 0 \).

In practice, we only need to show that if \( x_0 \in S \), then \( x_{n+1} \in S \), for \( S \) to be invariant.

**Cobweb diagram (graphic representation of maps)**

\[ x_{n+1} = f(x_n) \]

\[ y = x \]

\[ y = f(x) \]
\[ f'(x^*) < 0 \]

\[ x_{n+1} - x^* \approx f'(x^*) (x_n - x^*) \]

In higher dimensional maps, we look at \( \lambda_i(\text{D}f(x^*)) \).

- **Linearly stable:** if \( |\lambda_i(\text{D}f(x^*))| < 1 \) for all \( i \).
- **Linearly unstable:** if \( |\lambda_i(\text{D}f(x^*))| > 1 \) for some \( i \).

§ 6.2. Bifurcation of maps

\[ x_{n+1} = f_\mu(x_n) \]

**Criteria:**

\[ |\lambda_i(\text{D}f(x^*))| = 1 \]

**Saddle-node bifurcation:**

\[ x_{n+1} = x_n + \mu - x_n^2 = f_\mu(x_n) \]

- If \( \mu < 0 \), no fixed points.
- If \( \mu > 0 \), two fixed points \( x^\pm = \pm \sqrt{\mu} \).

One stable and one unstable, because

\[ f_\mu'(x^\pm) = 1 - 2x^\pm = 1 \mp 2\sqrt{\mu} \]

\( x^+_\mu = \sqrt{\mu} \) is stable and \( x^-_\mu = -\sqrt{\mu} \) is unstable.

\( x^* = \sqrt{\mu} \)
**Transcritical Bifurcation:**

\[ x_{n+1} = \mu (x_n) = x_n (1+\mu) x_n - x_n^2 = f_\mu(x_n) \]

Always two fixed points, \( x^* = 0 \) and \( x^* = \mu \).

\[ f_\mu'(x) = 1 + \mu - 2x \]

\[ f_\mu'(0) = 1 + \mu, \quad f_\mu'(\mu) = -1 + \mu \]

\( x^* = 0 \) is stable if \( \mu < 0 \) and unstable if \( \mu > 0 \) and \( x^* = \mu \) is stable if \( \mu > 0 \) and unstable if \( \mu < 0 \).

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**Pitchfork Bifurcation**

\[ x_{n+1} = (1+\mu) x_n - x_n^3 = f_\mu(x_n) \]

\( \mu < 0 \), one fixed point \( x^* = 0 \).

\( \mu > 0 \), three fixed points: \( x^* = 0, \pm \sqrt{\mu} \).

\[ f_\mu'(x^0) = 1 + \mu - 3x^2 \]

\( x^* = 0 \) is stable for \( \mu < 0 \), and unstable for \( \mu > 0 \).

\( x^* = \pm \sqrt{\mu} \) stable for \( \mu > 0 \), unstable for \( \mu < 0 \) and unstable for \( \mu > 0 \).