Exercise Sheet 10: Bifurcation of maps

1. Although the bifurcation diagrams of the continuous system

\[ \dot{x} = \mu x - x^2 \]

and the discrete map

\[ x_{n+1} = (1 + \mu)x_n - x_n^2 \]

look the same, the latter is more complicated at least in the following two ways:
(a) The solution \( x(t) \) for the continuous system usually converges to the fixed points for a large class of initial condition \( x(0) = x_0 \), but \( x_n \) for the discrete system may not (and usually converges for a limited range of initial condition). For instance, consider the case \( \mu = 1 \) and \( x_0 > 0 \), show that the solution \( x(t) \) to \( \dot{x} = \mu x - x^2 = x - x^2 \) converges to the stable fixed point \( x^* = 1 \). For the discrete system \( x_{n+1} = 2x_n - x_n^2 \), find the solution \( x_n \), by using the equivalent recursive relation \( x_{n+1} - 1 = -(x_n - 1)^2 \), and hence find for which \( x_0 > 0 \), the solution \( x_n \) does not converge to \( x^* = 1 \).

(b) The fixed point \( x^* = \mu \) (for \( \mu > 0 \)) is stable for the continuous system, but may lose its stability for the discrete system when \( \mu \) is large. Find the critical value \( \mu^* \), such that the fixed point \( x^* = \mu \) becomes unstable for \( x_{n+1} = (1 + \mu)x_n - x_n^2 \). What kind of bifurcation occurs at \( \mu = \mu^* \)?

2. For what values of \( \mu \), \([-2,0]\) is an invariant interval for \( x_{n+1} = \mu - x_n^2 \)? In other words, for which values of \( \mu \), if \( x_n \in [-2,0] \), then \( x_{n+1} \in [-2,0] \).

3. Consider the map

\[ x_{n+1} = G(x_n, \mu) = (1 + \mu)x_n + 4x_n^2 - 4x_n^3. \]

with \( x_n \in \mathbb{R} \) and \( \mu \in \mathbb{R} \).
(a) Find the fixed points of the map.
(b) If \( x^* \) is a fixed point of a map \( f : \mathbb{R} \rightarrow \mathbb{R} \), state the condition(s) that would lead you to expect that a bifurcation occurs at \( x^* \).
(c) Find the bifurcation points of the fixed points of the map \( G \) defined above.
(d) Sketch the bifurcation diagram for the fixed points of \( G \), showing the stability of the fixed points and the nature of the bifurcations.
4. (This question is taken the final exam for Year 2016-2017) Consider the cubic map \( x_{n+1} = f_\mu(x_n) = \mu x_n(1 - x_n^2) \) with the non-negative real parameter \( \mu \). Its bifurcation diagram is shown below.

(a) For which values of \( \mu \geq 0 \), the interval \([-1, 1]\) is invariant under this map?

(b) Find all fixed points of this cubic map.

(c) Find \( \mu_1 \) (the first dashed line in the figure) where the first bifurcation occurs.

(d) Find \( \mu_2 \) (the second dashed line in the figure) where the second bifurcation occurs.

(e) If the above cubic map is changed to \( x_{n+1} = g_\mu(x_n, x_{n-1}) = \mu x_n(1 - x_{n-1}^2) \) for \( \mu \geq 0 \), then the first bifurcation still occurs at \( \mu_1 \), but the second one is different from \( \mu_2 \). Find the second bifurcation point for this new map.

![Bifurcation Diagram](image.png)

**Figure 1.** The bifurcation diagram for the cubic map \( x_{n+1} = \mu x_n(1 - x_n^2) \) in Question 4.