1. The dynamic on the centre manifold usually could be simplified into different canonical forms, so that the qualitative behaviours are still preserved when the bifurcation parameter is changed. Simplify the following ODEs, by only keeping the dominant terms.
   (a) \( \dot{x} = \mu^2 - \mu^2 x - x^2 - \mu x^2 + \cdots \)
   (b) \( \dot{x} = \mu x - x^3 - \mu^2 x - \mu x^2 + \cdots \)
   (c) \( \dot{x} = \mu^2 - x + x^3 - \mu x + \cdots \)

2. Find the critical value of \( \mu^* \) for which there is a bifurcation at the origin for the system \( \ddot{x} = y - x - x^2, \quad \dot{y} = \mu x - y - y^2 \). Find the evolution equation on the extended centre manifold correct to third order terms and hence identify the bifurcation as a transcritical bifurcation. Hint: shift the parameter \( \mu \) to powers of \( \mu - \mu^* \) in the expansions/approximations.

3. A bead is free to move around the frictionless wire hoop, which is rotation at a fixed rate \( \omega \) around its vertical axis. If the bead’s position is specified by the angle \( \theta \), and the radius of the hoop is \( R \), then the equation for the motion is
   \[ \ddot{\theta} = \left( \omega^2 \cos \theta - \frac{g}{R} \right) \sin \theta. \]
   If the frequency \( \omega \) increases, a bifurcation occurs as the solution \( \theta = \dot{\theta} = 0 \) is no longer stable. Find the critical frequency \( \omega^* \) and classify the bifurcation.

![Figure 1. Bead on a rotating hoop for Question 3](image)
4. Consider the equation
\[ \dot{x} = x(y - 1) - az, \quad \dot{y} = 1 - x^2 - y, \quad \dot{z} = x - z. \]
Show that there is a bifurcation at \( a = 0 \) and use simple arguments to show that this is likely to be a pitchfork bifurcation. To confirm this
(a) Shift coordinates so that the bifurcation occurs at the origin in the new coordinates (since it is at \( a = 0 \) a shift in parameters is not needed).
(b) Bring the linear part into normal form.
(c) Find the extended centre manifold correct to third order.
(d) Describe the dynamics on the extended centre manifold for \( a < 0 \) and \( a > 0 \) (\( |a| \) small).

5. Given
\[ \dot{x} = -2x + 3y + \mu x + y^3, \quad \dot{y} = 2x - 3y - x^3 \]
(a) Explain why you would expect a bifurcation to occur at \( \mu = 0 \).
(b) Bring the equation into linear normal form at \( \mu = 0 \) and describe briefly the linear centre manifold.
(c) Consider the extended centre manifold for \( |\mu| \) small. Is a further change of coordinates necessary? If so, bring the extended system into normal form.
(d) Find the extended centre manifold up to and including the cubic terms and write down the leading order terms of the equation on the extended centre manifold.
(e) Describe the changes in behaviour of the system on the extended centre manifold as \( \mu \) passes through zero and hence identify the bifurcation.

6. As the parameter \( a \) increase from zero, a bifurcation occurs at \( a^* (> 0) \) for the system
\[ \dot{x} = -ax + y, \quad \dot{y} = \frac{x^2}{1 + x^2} - y, \]
because certain fixed points vanishes. Find this critical value \( a^* \), and the type of bifurcation (just based on the behaviour near the bifurcation point, no need to any coordinate transformation).

7. Show that the system
\[ \dot{x} - (\mu - x^2)\dot{x} + x = 0 \]
has a Hopf bifurcation if \( \mu = 0 \).