1. The Newton’s iteration \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) can be used to find the root of the equation \( f(x) = 0 \) efficiently in many situations. For example, for given constant \( a > 0 \), the square root \( \sqrt{a} \) can be obtained by setting \( f(x) = x^2 - a \), and the map used to find \( \sqrt{a} \) is then given by
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n^2 + a}{2x_n},
\]
with any initial condition \( x_0 > 0 \). We can show that \( x_n \) converges to \( \sqrt{a} \), as \( n \) goes to infinity as follows.

i) Regardless the value of \( x_0 > 0 \), show that \( x_n \geq \sqrt{a} \) for any \( n \geq 1 \).

ii) The sequence \( \{x_n\}_{n=1}^\infty \) is monotonically non-increasing. That is, \( x_{n+1} \leq x_n \) for any \( n \geq 1 \).

iii) The monotonically decreasing sequence \( \{x_n\}_{n=1}^\infty \) must converge to some limit \( x^* \geq \sqrt{a} \) (this is a basic fact about real numbers), i.e. \( \lim_{n \to \infty} x_n = x^* \). Therefore, \( x^* \) must be a fixed point of the map, because
\[
x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_n) = f(x^*).
\]

Finally since the only fixed point of the map \( x_{n+1} = f(x_n) \) not smaller than \( \sqrt{a} \) is \( \sqrt{a} \), we have \( x^* = \sqrt{a} \).

\[
\begin{align*}
x_0 &= 1.000000000000000 \\
x_1 &= 1.500000000000000 \\
x_2 &= 1.416666666666667 \\
x_3 &= 1.414215686274510 \\
x_4 &= 1.414213562374690 \\
x_5 &= 1.414213562373095 \\
\sqrt{2} &= 1.414213562373095
\end{align*}
\]

**Table 1.** Iteration of the map \( x_{n+1} = (x_n^2 + 2)/2x_n \) with \( x_0 = 1 \) to approximate \( \sqrt{2} \) and you can get all fourteen digits correct within five steps.

2. Solve the linear difference equation (the Fibonacci sequence)
\[
x_{n+1} = x_n + x_{n-1}
\]
with \( x_0 = x_1 = 1 \) in the following two ways.
(a) Assume that the solution takes the following form
\[ x_n = as_1^n + bs_2^n \]
where \( s_1 \) and \( s_2 \) are two roots of the quadratic equation \( s^2 = s + 1 \). Find \( s_1 \) and \( s_2 \) first, and then the two constants \( a \) and \( b \) from the initial conditions.

(b) The map is second order \((x_{n+1} \) depends on both \( x_n \) and \( x_{n+1} \), and can be reduced into a system of two first order maps, by introducing \( y_n = x_{n+1} \).
Alternatively, we can directly introduce the vector \( v_n = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} \), then
\[ v_{n+1} = Mv_n \]
for some constant matrix \( M \). Find \( M \) and then \( v_n = M^n v_0 \).

The matrix power \( M^n = \Lambda^n L^{-1} \), if \( M \) is diagonalized by \( L \), i.e., \( M = \Lambda L L^{-1} \).

3. Consider the map \( x_{n+1} = \mu x_n (1 - x_n^2) \). For which positive values of \( \mu \) is the interval \([-1, 1] \) invariant? That is, for which values of \( \mu \), if \( x_n \in [-1, 1] \), then \( x_{n+1} \) is also in the interval \([-1, 1] \).

4. Show that the parabola
\[ S = \{ (x, 4x^2) : x \in \mathbb{R} \} \]
is invariant under the map
\[ x_{n+1} = x_n/2, \quad y_{n+1} = 2y_n - 7x_n^2. \]
In fact, you only need to show that, if \((x_n, y_n)\) belongs to \( S \), then so does \((x_{n+1}, y_{n+1})\).

5. Find necessary and sufficient conditions on the trace and determinant of a \( 2 \times 2 \) real constant matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) that imply that its eigenvalues lie inside the (open) unit circle of the complex plane. If the eigenvalues are distinct, what does this imply about solutions to \( x_{n+1} = Ax_n \)?