MATH 44041/64041 Applied Dynamical Systems

Exercise Sheet 2: Picard iteration and Linearization

1. Use three Picard iterations with initial guess equal to a constant to obtain a polynomial approximation to the general solution of

\[ \dot{x} = 2x. \]

By solving the equation exactly find how many of these terms are correct in the power series expansion of solutions.

2. Use Picard iteration to find solutions of the equation

\[ \dot{x} = -5x + x^2, \quad x(0) = 1 \]

correct up to and including terms in \( t^3 \).

3. A simple population model for the number of rabbits \( r \) and sheep \( s \) on an isolated island is \( \dot{r} = r(2 - s - r), \dot{s} = s(3 - 2r - s) \) in appropriately normalised units. Find the stationary points and determine their type. Sketch the flow in the positive quadrant of the \( (r, s) \)-plane.

4. A simple population model for the number of rabbits \( r \) and eagles \( e \) on an isolated island is \( \dot{r} = r(2 - e - r), \dot{e} = e(-1 + r) \) in appropriately normalised units. Find the stationary points and determine their type. Sketch the flow in the positive quadrant of the \( (r, e) \)-plane.

5. The origin \( (0, 0) \) is a stationary point of the differential equation \( \dot{x} = f(x, y), \dot{y} = g(x, y) \) and the Jacobian matrix of right hand sides evaluated at the origin is

\[
\begin{pmatrix}
-3 & -1 \\
2 & 0
\end{pmatrix}
\]

Determine the linear type of the origin.

6. Find the stationary points and determine their type for the predator-prey model

\[ \dot{x} = x(-1 + y), \quad \dot{y} = y(4 - x - 2y). \]

Determine which is the predator and which the prey (and why?). Sketch a plausible phase portrait for the system and comment on the eventual outcome.
7. By working in polar coordinates, describe the local dynamics of the nonlinear stationary point at the origin for
\[ \dot{x} = xy - x^2y + y^3, \quad \dot{y} = y^2 + x^3 - xy^2. \]

8. Find the stationary points of the system
\[ \dot{x} = x^2 - y - 1, \quad \dot{y} = (x - 2)y \]
and identify their linear types. Show that the lines connecting the stationary points are invariant and hence sketch a plausible phase portrait for the system.  
**Hint:** The set \( H(x, y) = c \) is invariant if \( \dot{H} = 0 \) on \( H = c \).

9. Consider the Nosé equations \( \dot{x} = -y - xz, \dot{y} = x, \dot{z} = \alpha(x^2 - 1) \) where \( \alpha \) is a real parameter. Show that this system has no stationary points. Show that the \( z \)-axis is invariant and describe the dynamics on this line. If \( \alpha > 1 \) find two more invariant lines and describe the dynamics on these lines. (Hint: parametrize the lines as \( x \) and \( y \) being functions of \( z \).)