Exercise Sheet 1: Phase portrait, invariant sets, existence & Uniqueness

1. Suppose
\[ \dot{x} = x^2 + (2 - \mu)x - 2\mu \]
for some real parameter \( \mu \). By sketching the right hand side of this equation determine the behaviour of solutions as a function of \( \mu \).

2. Consider the logistic equation
\[ \dot{x} = \mu x (1 - x) \]
where \( \mu \) is a positive real parameter. Solve the equation for initial condition \( x_0 \geq 0 \) and describe the behaviour of solutions as \( t \to \infty \). Confirm your results by sketching the right hand side of the \( \dot{x} \) equation.

3. Does the exponential function \( x_0 e^{at} \), \( a \neq 0 \), have the semi-group property
\[ \phi_{t+s}(x_0) = \phi_s(\phi_t(x_0)), \quad t, s > 0? \]
What about the functions \( x_0 \sin \omega t \), \( x_0 + at \) and \( x_0 t \)?

4. Show that \( \varphi_t(x_0) = \frac{x_0}{\sqrt{1 + 2tx_0^2}} \) satisfies the semi-group property. Find a function \( f(x) \), such that \( x(t) = \varphi_t(x_0) \) is a solution to the first order autonomous ODE \( \dot{x} = f(x) \) with initial condition \( x(0) = x_0 \) (the initial condition is already satisfied).

5. Consider the differential equations
\[ \dot{x} = A - x - xy, \quad \dot{y} = -y + xy, \]
where \( A \) is a positive number. Find an equation for the time derivative of \( G(t) = x(t) + y(t) \) and hence show that if \( x(0) + y(0) = A \) then \( x(t) + y(t) = A \) for all \( t > 0 \).

6. (Lanchester’s laws of battle) Suppose \( A \) and \( B \) represent the number of soldiers in two armies engaging each other in a combat. The Lanchester’s laws of battle states that the number of soldiers is lost at a rate that is proportional to the number of soldiers on the other side. That is,
\[ \dot{A} = -\beta B, \quad \dot{B} = -\alpha A. \]
Find the conditions on the initial condition \( A_0, B_0 \) and the coefficients \( \alpha \) and \( \beta \) such that \( B \) loses the battle (\( B \) becomes zero earlier than \( A \)).
7. Consider the differential equation
\[ \dot{x} = x - 3y + 4xy, \quad \dot{y} = -x - y + 4x^2. \]
Show that the line \( y = x \) is invariant.

8. Consider
\[ \dot{x} = -x + x(x^2 + y^2), \quad \dot{y} = -y + y(x^2 + y^2). \]
Show that the circle \( x^2 + y^2 = 1 \) is invariant.

9. Consider the two dimensional system \( \dot{x} = Ax \) where \( A \) is a constant \( 2 \times 2 \) matrix. If \( A \) has a (simple) real eigenvalue \( \lambda \) with eigenvector \( e = (1, m)^T \) with \( m \in \mathbb{R} \) show that the line through the origin parallel to \( e \) is invariant.

10. (Generalisation of Question 9) If \( A \) is a constant real \( n \times n \) matrix and \( e \) is an eigenvector for the simple real eigenvalue \( \lambda \), show that the line through the origin parallel to \( e \) is invariant.

11. Consider the linear systems in the plane \( \dot{x} = Ax \) for each of the cases of \( A \) equal to
\[ \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix}. \]
Find a linear transformation that brings the system into (real) normal form and sketch solutions in the plane of these new coordinates and in the original coordinates.

12. Find necessary and sufficient conditions on the trace and determinant of a \( 2 \times 2 \) real constant matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) that imply that its eigenvalues have negative real parts. If the eigenvalues are distinct, what does this imply about solutions to \( \dot{x} = Ax \)?

13. Find the first three Picard’s iteration \( x^{(0)}(t), x^{(1)}(t), x^{(2)}(t) \) for the ODE
\[ \dot{x} = 1 - xt, \quad x(0) = 0. \]
Also find the first four coefficients in the series approximation \( x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \), by matching the initial condition and the ODE including and up to order \( O(t^2) \).