1. (a) The linearised system near the origin is
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
\]
The two eigenvalues are \(\lambda_1 = 1, \lambda_2 = -1\) with the eigenvectors
\[e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]
The transformation matrix is
\[
U = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]
and the transformation is
\[
\begin{pmatrix} u \\ v \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x + y)/2 \\ (y - x)/2 \end{pmatrix},
\]
with the inverse transform
\[
\begin{pmatrix} x \\ y \end{pmatrix} = [e_1 \ e_2] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ u + v \end{pmatrix}.
\]
The system in terms \(u\) and \(v\) is
\[
\dot{u} = \frac{\dot{x} + \dot{y}}{2} = \frac{y + x + x^2}{2} = u + \frac{(u - v)^2}{2},
\]
and
\[
\dot{v} = \frac{\dot{y} - \dot{x}}{2} = \frac{x^2 + x - y}{2} = -v + \frac{(u - v)^2}{2}.
\]
From the linearised system \(\dot{u} = u, \dot{v} = -v\), we get \(E^s = \{(u, v) \mid u = 0\}\) and the stable manifold \(W^s\) for the full system can be represented as
\[
W^s = \{(u, v) \mid u = h(v) = av^2 + bv^3 + \cdots \}.
\]
Then the condition that \(W^s\) is invariant implies \(0 = \frac{d}{dt}(u - h(v))|_{u=h(v)}\). Since
\[
\frac{du}{dt}|_{u=h(v)} = u + \left. \frac{(u - v)^2}{2} \right|_{u=h(v)} = av^2 + bv^3 + \cdots + \left(\frac{av^2 + bv^3 + \cdots - v^2}{2}\right) = \left(a + \frac{1}{2}\right)v^2 + (b-a)v^3 + \cdots \]
and
\[
\frac{d}{dt} h(v) \bigg|_{u=h(v)} = (2av + 3bv^2 + \cdots) \dot{v} = (2av + 3bv^2 + \cdots) \left(-v + \left(\frac{av^2 + bv^3 + \cdots - v^2}{2}\right)\right) = -2av^2 + (-3b + a)v^3 + \cdots,
\]
the invariance of \(W^s\) becomes
\[
\left(a + \frac{1}{2}\right)v^2 + (b-a)v^3 + \cdots = -2av^2 + (-3b + a)v^3 + \cdots.
\]
The matching conditions for the coefficients of \(v^2\) and \(v^3\) gives \(a + 1/2 = -2a, b - a = -3b + a\), or \(a = -1/6, b = -1/12\). Therefore, the stable manifold is
\[
W^s = \left\{(u, v) \mid u = -\frac{1}{6}v^2 - \frac{1}{12}v^3 + \cdots \right\}.
\]
From the condition that \( W \) the vector

\[
W = \{(x, y) \mid \frac{x + y}{2} = -\frac{1}{24}(y - x)^2 - \frac{1}{96}(y - x)^3 + \cdots \},
\]

(b) If \( W^s \) is represented by \( W^s = \{(x, y) \mid y = c_0 + c_1 x + c_2 x^2 + \cdots \} \). Then from the condition that \( W^s \) passes through the fixed point \((0, 0)\), we get \( y|_{x=0} = c_0 = 0 \) (matching the function value). From the condition that \( W^s \) is tangent to \( E^s = \{(x, y) \mid y = -x \} \) (the straight line spanned by the vector \( e_2 \)), we get \( y'(x)|_{y=0} = c_1 = -1 \) (matching the slope). Finally, the coefficient \( c_2 \) is determined from the invariance of \( W^s = \{(x, y) \mid y = -x + c_2 x^2 + \cdots \} \), that is

\[
0 = \frac{d}{dt}(y + x - c_2 x^2 + \cdots)|_{y=-x+c_2 x^2+\cdots}
= (\dot{y} + (1 - 2c_2 x + \cdots)\dot{x})|_{y=-x+c_2 x^2+\cdots}
= (x^2 + x + (1 - 2c_2 x + \cdots)(-x + c_2 x^2 + \cdots))
= (1 + 3c_2)x^2 + \cdots
\]
or \( c_3 = -1/3 \). That is, the stable manifold is represented by \( y = -x - x^2/3 + \cdots \).

(c) Using Taylor expansion,

\[
y = -x \sqrt{1 + 2x/3} = -x \left(1 + \frac{1}{3}x - \frac{1}{18}x^2 + \cdots\right),
\]

which is consistent with (b). Substituting the change of variable \( x = u - v \), \( y = u + v \) into the expression \( y = -x \sqrt{1 + 2x/3} \) and assuming \( u = h(v) = av^2 + bv^3 + \cdots \), we get

\[
(\dot{v}^2 + bv^3 + \cdots) + v = -(av^2 + bv^3 + \cdots - v)^2/3 + 2(\dot{v}^2 + bv^3 + \cdots - v)/3
= -(av^2 + bv^3 + \cdots - v) \left(1 - \frac{v}{3} + \left(\frac{a}{3} - \frac{1}{18}\right)v^2 + \cdots\right).
\]

Matching the coefficients of \( v^2 \) and \( v^3 \) on both sides, we still get \( a = -1/6, b = -1/12 \), still consistent with (a).

Remark: The same manifold can be represented in many different ways. But if the linearised system is not in diagonal form, the condition that \( E^s \) (or \( E^u \)) is tangent to \( W^s \) (or \( W^u \)) has to be used explicitly to determined the constant and linear coefficients. Otherwise if the linearised system is in diagonal form, you can start with quadratic and higher order terms. However, if we are interested in the dynamics on the centre manifold (or any possible bifurcation), the change of variables into normal form (for the linearised system) is essential.

2. From the eigenvector \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) associated with the positive eigenvalue 1, we know that \( E^u \) is a straight line passing through \((-1, -1)\) with slope 1/2. Therefore

\[
E^u = \{(x, y) \mid y + 1 = (x + 1)/2\}.
\]

Then the unstable manifold \( W^u \) of the full system can be parametrised by

\[
W^u = \{(x, y) \mid y + 1 = (x + 1)/2 + c_2(x + 1)^2 + c_3(x + 1)^3 + \cdots\}.
\]

By writing the system \( \dot{x} = -xy, \dot{y} = x - y \) in terms of the factors \( x + 1 \) and \( y + 1 \), i.e.,

\[
\dot{x} = (x + 1) - (x + 1)(y + 1), \quad \dot{y} = (x + 1) - (y + 1),
\]

we can find the coefficients \( c_2, c_3, \cdots \) using the condition that \( W^u \) is invariant. That is,

\[
\frac{d}{dt}(y + 1)\bigg|_{y+1=(x+1)/2+c_2(x+1)^2+\cdots} = \frac{d}{dt}( (x + 1)/2 + c_2(x + 1)^2 + \cdots )\bigg|_{y+1=(x+1)/2+c_2(x+1)^2+\cdots}.
\]
3. (a) From $\dot{x} = y - x^2 = \frac{\partial H(x,y)}{\partial y}$, we can integrate both sides with respect to $x$ to get

$$H(x,y) = xy - \frac{1}{3}x^3 + F(y),$$

for some function $F$. Then using the first equation

$$\dot{x} = -x + y^2 = -\frac{\partial H(x,y)}{\partial y} = -x - F'(y).$$

Therefore, $F'(y) = -y^2$ and $F(y) = -y^3/3$ (forget about the integration constant), or $H(x,y) = xy - (x^3 + y^3)/3$.

(b) Since

$$\dot{H} = \dot{x}\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} = (-x + y^2)(y - x^2) + (y - x^2)(x - y^2) = 0,$$

$H(x,y)$ does not change along the trajectory and the level curves $S_c$ are invariant.

(c) Since the linearised system is $\dot{x} = -x, \dot{y} = y$, we have

$$E^s = \{(x,y) \mid y = 0\}, \quad E^u = \{(x,y) \mid x = 0\}.$$

The stable manifold $W^s$ is $\{(x,y) \mid y = ax^2 + bx^3 + \cdots\}$. Since

$$\dot{y} = y - x^2 = (a - 1)x^2 + bx^3 + \cdots$$

and

$$\dot{y} = (2ax + 3bx^2 + \cdots)\dot{x} = (2ax + 3bx^3 + \cdots)(-x + y^2) = (2ax + 3bx^2 + \cdots)(-x + (ax^2 + bx^3 + \cdots)^2) = -2ax^2 - 3bx^3 + \cdots.$$

Therefore, $c_2$ is determined by the matching condition of the coefficients of the quadratic terms on both sides. That is $-c_2 = 2c_2 - 1/4$, or $c_2 = 1/12$.

**Remark:** Because the fixed point $(-1, -1)$ is different from the origin, any approximation near this fixed point should be written as powers of $x + 1$ and $y + 1$. Equivalently, we can shift the coordinates to $u = x + 1$ and $v = y + 1$, and work on powers of $u$ and $v$ as most other problems with fixed points at the origin.
Matching the coefficients of $x^2$ and $x^3$ on both sides, we get $a = 1/3$ and $b = 0$ and $y = x^2/3 + 0x^3 + \cdots$.

The unstable manifold is $\{(x, y) \mid x = cy^2 + dy^3 + \cdots\}$. From

$$\dot{x} = -x + y^2 = (1-c)y^2 - dy^3 + \cdots$$

and

$$\dot{x} = (2cy + 3dy^2 + \cdots)\dot{y} = (2cy + 3dy^2 + \cdots)(y - (cy^2 + dy^3 + \cdots)^2) = 2cy^2 + 3dy^3 + \cdots.$$  

Matching the coefficients of $y^2$ and $y^3$, we get $c = 1/3$ and $d = 0$ and $x = y^2/3 + 0y^3 + \cdots$.

4. For the system $\dot{x} = -2x + y - x^2$, \quad $\dot{y} = x(y - x)$ near the origin, the matrix for the linear part is  

$$\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$  

with

$$\lambda_1 = 0, \quad e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

From the eigenvectors, we can find the coordinate transformation matrix

$$U = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}.$$  

As a result, the change of variable is $u = \frac{1}{2}y$, $v = x - \frac{1}{2}y$, with the inverse transformation $y = 2u$, $x = u + v$. Thus

$$\begin{array}{rcl}
\dot{u} & = & \frac{1}{2}\dot{y} = \frac{1}{2}((u + v)(u - v) - \frac{1}{2}(u^2 - v^2)) \\
\dot{v} & = & \dot{x} - \frac{1}{2}\dot{y} \\
& = & -2v - (u + v)^2 - \frac{1}{2}(u^2 - v^2) \\
& = & -2v - (\frac{1}{2}u^2 + 2uv + \frac{3}{4}u^2).
\end{array}$$  

Since $E^c = \{(u, v) \mid v = 0\}$. the power series for the centre manifold tangential to this:

$$v = au^2 + bu^3 + \ldots.$$  

Then

$$\dot{v} = -2(au^2 + bu^3 + \ldots) - ((au^2 + bu^3 + \cdots) + 2au^3 + \cdots + \frac{3}{2}u^2)$$  

and by differentiating the equation for the invariant manifold

$$\dot{v} = \dot{u}(2au + 3bu^2 + \ldots) = au^2 + \cdots.$$  

Equating terms of order $u^2$ between these two representations of $\dot{v}$ gives

$$-2a - \frac{3}{2} = 0, \quad \text{so} \quad a = -\frac{3}{4}.$$  

and (although not needed) at order $u^3$

$$-2b - 2a = a, \quad \text{so} \quad b = \frac{9}{8}.$$  

Thus the centre manifold is $v = -\frac{3}{4}u^2 + \frac{9}{8}u^3 + \ldots$.

Thus the equation on the centre manifold is

$$\dot{u} = \frac{1}{2}u^2 - \frac{1}{2}\left( -\frac{3}{4}u^2 + \frac{9}{8}u^3 + \ldots \right)^2 = \frac{1}{2}u^2 - \frac{9}{32}u^4 + \cdots.$$  

From the leading order $\dot{u} \approx u^2/2$, $u$ converges to the origin if $u < 0$ and $u$ escapes away from the origin if $u > 0$. 

The condition which quadratic or higher order in $x$ and $y$ make sure that you select a small domain (say $x \in [-1, 1], y \in [-1, 1]$), because there is a fixed point at $(-1, 1)$ that changes the local behaviours.

5. The linearised system near the origin is

$$
\begin{pmatrix}
x
y
z
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
x
y
z
\end{pmatrix}
$$

whose eigenvalues are governed by the characteristic equation

$$
\det \begin{pmatrix}
\lambda & 1 & 0 \\
-1 & \lambda & 0 \\
0 & 0 & \lambda + 1
\end{pmatrix} = (\lambda + 1) \det \begin{pmatrix}
\lambda & 1 \\
-1 & \lambda
\end{pmatrix} = (\lambda + 1)(\lambda^2 + 1),
$$

or $\lambda_1 = -1, \lambda_2 = i, \lambda_3 = -i$. Therefore, $E^c = \{(x, y, z) \mid z = 0\}$ and the centre manifold $W^c$ for the full system can be expressed as $W^c = \{(x, y, z) \mid z = h(x, y)\}$ for some function $h(x, y)$ which quadratic or higher order in $x$ and $y$. Let $z = h(x, y) = ax^2 + bxy + cy^2 + \cdots$, then

$$
\frac{d}{dt}(z - h(x, y))\bigg|_{z=h(x,y)} = 0.
$$

From

$$
\frac{d}{dt} h(x, y)\bigg|_{z=h(x,y)} = (2ax + by)\dot{x} + (bx + cy)\dot{y}
$$

and

$$
\frac{d}{dt} h(x, y)\bigg|_{z=h(x,y)} = (2ax + by)(-y + yz + (y-z)(x^2 + y^2)) + (bx + cy)(x - xz - (x+y)(x^2 + y^2))
$$

the condition $\frac{d}{dt}(z - h(x, y))\bigg|_{z=h(x,y)} = 0$ becomes

$$
(1 - a)x^2 - bxy + (1 - c)y^2 + O(|x|^3, |y|^3) = bx^2 + (-2a + 2c)xy - by^2 + O(|x|^3, |y|^3),
$$

or

$$
1 - a = b, \quad -b = -2a + 2c, \quad 1 - c = -b.
$$

The solution is $a = c = 1, b = 0$. Therefore, the centre manifold at the origin is $z = x^2 + y^2 + \cdots$.

Substitute this representation into the equations for $x$ and $y$, we get

$$
\dot{x} = -y + yz + (y - z)(x^2 + y^2) = -y + y(x^2 + y^2 + \cdots) + (y - x^2 - y^2)(x^2 + y^2) = -y + 2y(x^2 + y^2) + \cdots
$$

and

$$
\dot{y} = x - xz - (x+y)(x^2 + y^2) = x - x(x^2 + y^2 + \cdots) - (x+2y)(x^2 + y^2) + \cdots = x - (2x+y)(x^2 + y^2) + \cdots.
$$

The equivalent system in terms of polar coordinates is

$$
\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-y^2(x^2 + y^2) + \cdots}{r} = -r^2 \sin^2 \theta + \cdots
$$

Remark: There is no need to go back the original variables $(x, y)$ (adding little insights). From the conclusion above, we can deduce that if $y(=2u) < 0$, then $(x, y)$ converges to the origin; if $y > 0$, then $(x, y)$ escapes to infinity. Use pplane to verify this by looking at the trajectories, and make sure that you select a small domain (say $x \in [-1, 1], y \in [-1, 1]$), because there is a fixed point at $(-1, 1)$ that changes the local behaviours.
\[ \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1 - (2x^2 + xy + 2y^2 + \cdots) = 1 - r^2(2\cos^2 \theta + \sin \theta \cos \theta + 2\sin^2 \theta) + \cdots. \]

Since \( \dot{r} = -r^3 \sin^2 \theta + \cdots \leq 0 \) for small and positive \( r \), \( r = 0 \) is a stable fixed point and the origin is stable.

**Remark:** Because \( \dot{\theta} = 1 + \cdots > 0 \), \( \theta \) will increase at the speed about 1, and \( \dot{r} = -r^3 \sin^2 \theta + \cdots \) will not stagnate on the lines \( \theta = 0 \) or \( \theta = \pi \). But this is possible for other systems like the following one
\[ \dot{x} = -xy^2 + y^2, \quad \dot{y} = -y^3 - xy, \]

or in the polar coordinates \( \dot{r} = -r^3 \sin^2 \theta, \dot{\theta} = -r \sin \theta \). Here the solution starting from a point \((x_0, y_0)\) close to the origin goes to \((\sqrt{x_0^2 + y_0^2}, 0)\) instead of the origin (stops moving once the point reaches \( \theta = 0 \), which is different from the case above).