1. (a) From \( r^2 = x^2 + y^2 \), we get \( r \dot{r} = x \dot{x} + y \dot{y} \) or
\[
\dot{r} = \frac{x \dot{x} + y \dot{y}}{r} = \frac{1}{r} [xy - x^2(x^2 + y^2 - 2x - 3) - xy - y^2(x^2 + y^2 - 2x - 3)]
\[
= -\frac{(x^2 + y^2)(x^2 + y^2 - 2x - 3)}{r} = r(3 + 2r \cos \theta - r^2).
\]

From \( \theta = \arctan(y/x) \),
\[
\dot{\theta} = \frac{y \dot{x} - x \dot{y}}{x^2 + y^2} = \frac{1}{r^2} [x(-x - y(x^2 + y^2 - 2x - 3)) - y(y - x(x^2 + y^2 - 2x - 3))] = -1.
\]

Therefore, \( \dot{r} = rf(r, \theta) \) with \( f(r, \theta) = 3 + 2r \cos \theta - r^2 \).

(b) The fixed points \((x, y)\) satisfies
\[
y = x(x^2 + y^2 - 2x - 3), \quad x = -y(x^2 + y^2 - 2x - 3).
\]

Therefore,
\[
x^2 + y^2 = x(-y(x^2 + y^2 - 2x - 3)) + y(x^2 + y^2 - 2x - 3) = 0,
\]
and the origin is the only fixed point (which is an unstable focus).

(c) From the expression \( f(r, \theta) = 3 + 2r \cos \theta - r^2 \) in (a), since \( r \geq 0 \),
\[
\max_{\theta} f(r, \theta) = f(r, 0) = 3 + 2r - r^2 = (3 - r)(1 + r), \quad \min_{\theta} f(r, \theta) = f(r, \pi) = 3 - 2r - r^2 = (3 + r)(1 - r).
\]

(d) If \( r > 3 \), then
\[
\dot{r} = rf(r, \theta) \leq r \max_{\theta} f(r, \theta) = r(3 - r)(1 + r) < 0.
\]

If \( 0 < r < 1 \), then
\[
\dot{r} = rf(r, \theta) \geq r \min_{\theta} f(r, \theta) = r(1 - r)(3 + r) > 0.
\]

(e) Consider the region
\[
D_\epsilon = \{(x, y) \mid 1 - \epsilon < \sqrt{x^2 + y^2} < 3 + \epsilon\} = \{(r, \theta) \mid 1 - \epsilon < r < 3 + \epsilon\}.
\]

Since \( \dot{r} < 0 \) on the outer boundary \( r = 3 + \epsilon \) and \( \dot{r} > 0 \) on the inner boundary \( r = 1 - \epsilon \), \( D_\epsilon \) is invariant for any \( \epsilon > 0 \). Moreover, there is no stationary point on \( D_\epsilon \). Then by Poincaré-Bendixson Theorem, there is at least one periodic orbit in the region \( D_\epsilon \). Since \( \epsilon \) can be any positive number, there is at least one periodic orbit on \( D_0 = \bigcap_{\epsilon > 0} D_\epsilon \).

2. If \( V(x, y) = x^2 + Ay^2 \), then \( V \geq 0 \) and
\[
\frac{1}{2} \dot{V} = x \dot{x} + Ay \dot{y}
\[
= x(x - 9y - x(x^2 + 9y^2)) + Ay(x + y - y(x^2 + 9y^2))
\[
= x^2 + Ay^2 + (A - 9)xy - (x^2 + Ay^2)(x^2 + 9y^2).
\]
Therefore, it makes sense to kill the term \( xy \) by choosing \( A = 9 \) and hence \( \frac{1}{2} V = (x^2 + 9y^2)(1 - (x^2 + 9y^2)) \).

If we choose the region

\[
D = \{(x, y) \mid 1/2 \leq x^2 + 9y^2 \leq 2\},
\]

then \( D \) is invariant. In fact, if \((x, y)\) is on the outer boundary \( \partial D_+ \), or \( x^2 + 9y^2 = 2 \), then \( \frac{1}{2} V = 2(1 - 2) = -2 < 0 \), which implies that \((x, y)\) moves into the region \( D \); if \((x, y)\) is on the inner boundary \( \partial D_- \), or \( x^2 + 9y^2 = 1/2 \), then \( \frac{1}{2} V = (1 - 1/2)/2 = 1/4 > 0 \), which implies that \((x, y)\) leaves the set \( \{(x, y) \mid x^2 + 9y^2 \leq 1\} \) and moves into \( D \).

On the other hand, if \((x, y)\) is a fixed point, then

\[
x - 9y - x(x^2 + 9y^2) = 0, \quad x + y(x^2 + 9y^2) = 0.
\]

Obviously, \((0, 0)\) is a fixed point. If there is another fixed point \((x, y)\) that is not at the origin, then

\[
0 = x(x - 9y - x(x^2 + 9y^2)) + 9y(x + y - y(x^2 + 9y^2)) = (x^2 + 9y^2)(1 - x^2 - 9y^2),
\]

and \( 1 - x^2 - 9y^2 = 0 \) (because \( x^2 + 9y^2 \neq 0 \)). As a result, the equations that characterise the fixed point are reduced into

\[
0 = x - 9y - x(x^2 + 9y^2) = x - 9y - x = 9y, \quad 0 = x + y(x^2 + 9y^2) = x + y - y = x,
\]

which gives the trivial solution \( x = y = 0 \), contradicting the assumption \( x^2 + 9y^2 = 1 \). Therefore, \((0, 0)\) is the only fixed point, which is not inside \( D \). According to Poincare-Bendixson Theorem, there exists a periodic solution inside \( D \).

3. Consider the system

\[
\dot{x} = 4y, \quad \dot{y} = -x + y - x^2y - 4y^3.
\]

Stationary points have \( y = 0 \) (from \( \dot{x} = 0 \)) and hence \( x = 0 \), so the only stationary point is \((0, 0)\) and the Jacobian is

\[
\begin{pmatrix}
0 & 4 \\
-1 & 1
\end{pmatrix}
\]

with characteristic equation \( s(s - 1) + 4 = 0 \), so \( s = (1 \pm i \sqrt{5})/2 \), so it is an unstable focus and there is a small closed curve surrounding it which trajectories cross outwards using the linearised Lyapunov function in reverse time.

For an outer boundary of a Poincaré-Bendixson region let \( V(x) = x^2 + 4y^2 \). Then

\[
\frac{1}{2} V = 4y^2(1 - x^2 - 4y^2)
\]

so if \( x^2 + 4y^2 > 1 \) then \( V \leq 0 \) (note we cannot make it strictly less than zero). Set \( V(x, y) = 2 \) to avoid the boundary case, though the argument works perfectly well there. Then \( V \leq 0 \) for all \( V \geq 2 \) and so if \((x_0, y_0)\) lies in \( v \leq 2 \) then using the Mean Value Theorem \( V(\phi(x_0, y_0, t)) \leq 2 \) for all \( t \geq 0 \) (suppose not, then there is a time \( t_1 \) at which \( V(\phi(x_0, y_0, t)) = 2 \) and \( \delta > 0 \) such that \( V(\phi(x_0, y_0, t)) > 2 \) if \( t \in (t_1, t_1 + \delta) \) and hence for \( t \) in this open region \( V(\phi(x_0, y_0, t)) - V(\phi(x_0, y_0, t_1)) > 0 \) and \( V(\phi(x_0, y_0, t)) - V(\phi(x_0, y_0, t_1)) = V(\xi)(t - t_1) \) for some \( \xi \) outside \( V = 2 \), a contradiction). Hence we have a Poincaré-Bendixson region (between \( V = 2 \) and the small curve containing the origin, hence there exists at least one periodic orbit.

4. We can solve the second equation \( \dot{y} = h(t)y \) which is separable and the solution is given by

\[
\ln(y(t)/y_0) = \int_{y_0}^{y(t)} \frac{dy}{y} = \int_0^t h(\tau)d\tau = \ln(2 + \sin \tau + \cos \tau) \bigg|_0^t = \ln \frac{2 + \sin t + \cos t}{3},
\]
or
\[ y(t) = \frac{2 + \sin t + \cos t}{3} y_0. \]

Then the first equation becomes
\[ \dot{x} = x + \frac{2 + \sin t + \cos t}{3} y_0, \]
which is linear that can be integrated with the integrating factor $e^{-t}$. The solution is obtained by integrating both sides of
\[ \frac{d}{dt} xe^{-t} = (\dot{x} - x)e^{-t} = \frac{2 + \sin t + \cos t}{3} y_0 e^{-t}, \]
or
\[ x(t) = e^t x_0 + e^t y_0 \int_0^t \frac{2 + \sin \tau + \cos \tau}{3} e^{-\tau} d\tau. \]

By writing the solution in the matrix form
\[ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \]
we can get the fundamental matrix $\Phi(t)$ as
\[ \Phi(t) = \begin{pmatrix} e^t & e^t \int_0^t \frac{2 + \sin \tau + \cos \tau}{3} e^{-\tau} d\tau \\ 0 & e^t \int_0^t \frac{2 + \sin \tau + \cos \tau}{3} e^{-\tau} d\tau \end{pmatrix}. \]

Since the (minimal) period is $T = 2\pi$, the monodromy matrix
\[ B = \Phi(T) = \begin{pmatrix} e^{2\pi} & e^{2\pi} \int_0^{2\pi} \frac{2 + \sin \tau + \cos \tau}{3} e^{-\tau} d\tau \\ 0 & e^{2\pi} \int_0^{2\pi} \frac{2 + \sin \tau + \cos \tau}{3} e^{-\tau} d\tau \end{pmatrix} \]
with two eigenvalues $\rho_1 = e^{2\pi}$ and $\rho_2 = 1$ (the characteristic multipliers). Therefore, the characteristic exponents are
\[ \mu_1 = \frac{1}{2\pi} \ln \rho_1 = 1, \quad \mu_2 = \frac{1}{2\pi} \ln \rho_2 = 0. \]