MATH 44041/64041 Applied Dynamical Systems

Answers to Exercise Sheet 4 : Nonlinear systems and stable/unstable manifolds

1. The trajectories are governed by the ODE
   \[
   \frac{dy}{dx} = \frac{y(2x - y)}{x^2}.
   \]
   This ODE is homogeneous (the right hand side does not change if \(x\) and \(y\) are replaced by \(\lambda x\) and \(\lambda y\) respectively), and can be solved by the change of variable \(y = zx\). The resulting ODE becomes
   \[
   \frac{dy}{dx} = \frac{d(xz)}{dx} = z + x \frac{dz}{dx} = z(2 - z),
   \]
   or equivalently
   \[
   x \frac{dz}{dx} = z(1 - z).
   \]
   This ODE is separable. Integrating both sides of
   \[
   \frac{dz}{z(1 - z)} = \frac{dx}{x},
   \]
   we get
   \[
   \ln x = \int \frac{dz}{x} = \int \frac{dz}{z(1 - z)} = \int \left( \frac{1}{z} - \frac{1}{z - 1} \right) dz = \ln \frac{z}{z - 1} + C
   \]
   for some constant \(C\). This equation can be simplified as
   \[
   x = k \frac{z}{z - 1},
   \]
   for some constant \(k\). Substituting \(z = y/x\) back, we get the equation for the trajectories
   \[
   x = k \frac{y}{y - x}.
   \]
   The trajectories are plotted in Figure 3.20 in the lecture notes.

   Remark: The linearised system \(\dot{x} = 0, \dot{y} = 0\) does not tell you anything, with two zero eigenvalues. However, we may still get some information by various ways (though not always work), for instance, by looking at the equations for the trajectories, or by transforming into polar coordinates.

2. First we show that \(W^s\) (or \(W^u\)) is tangent to \(E^s\) (or \(E^u\)). The linearised system is
   \[
   \begin{pmatrix}
   \dot{x} \\
   \dot{y}
   \end{pmatrix} = \begin{pmatrix}
   -1 & 0 \\
   1 & 1
   \end{pmatrix} \begin{pmatrix}
   x \\
   y
   \end{pmatrix}.
   \]
   The two eigenvalues are \(\lambda_1 = -1, \lambda_2 = 1\) and the corresponding eigenvectors are
   \[
   e_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
   \]
   Therefore, for this linearised system, the stable manifold is \(y = -x/2\) (this straight line has the same direction as \(e_1\); it is the stable manifold \(E^s\) because the associated eigenvalue \(\lambda_1\) is negative), and the unstable manifold \(E^u\) is \(x = 0\) (the y-axis), which coincides with \(W^u\). Since the line \(y = -x/2\) is tangent to \(y = -x/2 - x^2/3\) (because both pass the origin, and have the same slope \(-1/2\) at the origin), \(E^s\) is tangent to \(W^s\).
Next we show that both $W^s$ and $W^u$ are invariant sets. Define $G_1(x, y) = x$, then if $(x, y)$ is on the $y$-axis, that is $G_1(x, y) = x = 0$, then $\dot{G}_1 = \dot{x} = -x = 0$. That is the $y$-axis is invariant.

Define $G_2(x, y) = y + x/2 + x^2/3$, then if $(x, y)$ on $W^s$ or $G_2(x, y) = 0$, then

$$\dot{G}_2(x, y)|_{G_2(x, y) = 0} = \dot{y} + \left(\frac{1}{2} + \frac{2x}{3}\right) \dot{x} = y + x + x^2 - \left(\frac{1}{2} + \frac{2x}{3}\right) x = y + \frac{x}{2} + \frac{x^2}{3} = G_2(x, y) = 0.$$ 

(Alternative approach using the definition) The general solution for the system can be obtained exactly. From the first equation $\dot{x} = -x$, we get $x(t) = x_0 e^{-t}$, and the second equation becomes

$$\dot{y} = y + x + x^2 = y + x_0 e^{-t} + x_0^2 e^{-2t},$$

which is a linear ODE. Multiplying both sides with the integrating factor $e^{-t}$, we get

$$\frac{d}{dt}(ye^{-t}) = e^{-t}(\dot{y} - y) = x_0 e^{-2t} + x_0^2 e^{-3t}.$$ 

That is,

$$y(t)e^{-t} = y_0 + \int_0^t (x_0 e^{-2\tau} + x_0^2 e^{-3\tau}) d\tau = y_0 + \frac{x_0}{2}(1 - e^{-2t}) + \frac{x_0^2}{3}(1 - e^{-3t})$$

or

$$y(t) = y_0 e^t + \frac{x_0}{2}(e^t - e^{-t}) + \frac{x_0^2}{3}(e^t - e^{-2t}).$$

Therefore, if $(x_0, y_0) \in W^u = \{(x, y) | x = 0\}$, or equivalently $x_0 = 0$, the solution is $x(t) \equiv 0$ and $y(t) = y_0 e^t$. That is, $(x(t), y(t)) \in W^u$ and $W^u$ is invariant. Finally, since $(x(t), y(t)) \to (0, 0)$ as $t \to -\infty$, $W^u$ is the unstable manifold.

Similarly, if $(x_0, y_0) \in W^s = \{(x, y) | y + x/2 + x^2/3 = 0\}$, or $y_0 + x_0/2 + x_0^2/3 = 0$, then

$$y(t) = y_0 e^t + \frac{x_0}{2}(e^t - e^{-t}) + \frac{x_0^2}{3}(e^t - e^{-2t})$$

and

$$y(t) + \frac{x(t)}{2} + \frac{x(t)^2}{3} = -\left(\frac{x_0}{2} e^{-t} + \frac{x_0^2}{3} e^{-2t}\right) + \frac{x_0 e^{-t}}{2} + \frac{x_0^2 e^{-2t}}{3} = 0.$$ 

That is the solution $(x(t), y(t)) \in W^s$ and $W^s$ is invariant. Since $(x(t), y(t)) \equiv (x_0 e^{-t}, -(x_0 e^{-t}/2 + x_0^2 e^{-2t}/3))$ converges to zero, as $t$ goes to infinity, $W^s$ is the stable manifold.

3. From the governing equation, we get $y = \pm \sqrt{x^2 + \frac{2}{3}x^3}$. We can choose the right signs for the stable/unstable manifolds from the linearized system

$$\dot{x} = y, \quad \dot{y} = x^2 + x.$$ 

The two eigenvalues of the coefficient matrix are roots of

$$\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 - 1,$$

or $\lambda_\pm = \pm 1$. The two eigenvalues are

$$e_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
and the straight lines corresponding to \( e_\pm \) are \( y = \pm x \). Therefore, the unstable manifold \( W^u \) represented by \( y = M_+(x) \) is tangent to \( y = x \), implying that near the origin \( M_+(x) \) is positive for \( x > 0 \) and is negative for \( x < 0 \). Therefore,

\[
M_+(x) = \begin{cases} \sqrt{x^2 + \frac{2}{3}x^3}, & x > 0, \\ -\sqrt{x^2 + \frac{2}{3}x^3}, & x < 0. \end{cases}
\]

This can be written in a more compact form as \( y = M_+(x) = x\sqrt{1 + 2x/3} \). Similarly, for the stable manifold, \( y = M_-(x) = -x\sqrt{1 + 2x/3} \).

4. **Approach based on the geometric characterisation: \( W^s \) is invariant and tangent to \( E^s \)** The linearised system near the origin is \( \dot{x} = -x, \dot{y} = y + g'(0)x \). The two eigenvalues of the coefficient matrix \( \begin{pmatrix} -1 & 0 \\ g'(0) & 1 \end{pmatrix} \) are \( \lambda_\pm = \pm 1 \), together with the two eigenvectors

\[
\vec{e}_- = \begin{pmatrix} 1 \\ -g'(0)/2 \end{pmatrix}, \quad \vec{e}_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Therefore, the stable manifold \( W^s \) for the linearised system is the straight line \( \{ (x, y) \mid y = -g'(0)/2 \} \).

Because \( g(0) = 0 \), the curve \( y = -\int_0^1 g(xu)du \) pass through the origin \((0, 0)\), with a slope

\[
\frac{d}{dx} \left( -\int_0^1 g(xu)du \right) \bigg|_{x=0} = -\int_0^1 u g'(xu)du \bigg|_{x=0} = -\int_0^1 u g'(0)du = -g'(0)/2.
\]

Therefore, the curve \( y = -\int_0^1 g(xu)du \) is tangent to \( E^s \). Next we can show that the set \( W^s = \{ (x, y) \mid y = -\int_0^1 g(xu)du \} \) is invariant. Define \( G(x, y) = y + \int_0^1 g(xu)du \), then

\[
\frac{d}{dt} G(x, y) \bigg|_{G(x, y)=0} = \dot{y} + \dot{x} \int_0^1 u g'(xu)du \bigg|_{G(x, y)=0} = y + g(x) - x \int_0^1 u g'(xu)du \bigg|_{G(x, y)=0} = g(x) - \int_0^1 g(xu)du - x \int_0^1 u g'(xu)du.
\]

Since \( \int_0^1 g(xu)du + x \int_0^1 u g'(xu)du = \int_0^1 \frac{d}{du}(ug(xu))du = ug(xu) \bigg|_{u=0} = g(x), \frac{d}{dt} G(x, y) \bigg|_{G(x, y)=0} = 0 \) and \( W^s = \{ (x, y) \mid G(x, y) = 0 \} \) is invariant.

(\textit{Approach based on the definition of stable manifold}) From the first equation \( \dot{x} = -x \), we get \( x(t) = x_0 e^{-t} \) and the second equation becomes \( \dot{y} = y + g(x_0 e^{-t}) \). This linear ODE can be solved with the integrating factor \( e^{-t} \), so that

\[
\frac{d}{dt} (e^{-t} y) = e^{-t}(\dot{y} - y) = e^{-t} g(x_0 e^{-t}).
\]

Therefore, \( y(t)e^{-t} = y_0 + \int_0^t e^{-\tau} g(x_0 e^{-\tau})d\tau \), or \( y(t) = y_0 e^t + e^t \int_0^t e^{-\tau} g(x_0 e^{-\tau})d\tau \). Now if \( (x_0, y_0) \in \{ (x, y) \mid y = -\int_0^1 g(xu)du \} \), or \( y_0 = -\int_0^1 g(x_0 u)du \), then

\[
y(t) + \int_0^1 g(x(t)u)du = y_0 e^t + e^t \int_0^t e^{-\tau} g(x_0 e^{-\tau})d\tau + \int_0^1 g(x_0 e^{-t}u)du
\]

\[
= e^t \left[ -\int_0^1 g(x_0 u)du + \int_0^t e^{-\tau} g(x_0 e^{-\tau})d\tau + e^{-t} \int_0^1 g(x_0 e^{-t}u)du \right]
\]
If the change of variable $u = e^{-\tau}$ is used in the second integral, then $du = -e^{-\tau}d\tau$, and
$$
\int_0^t e^{-\tau} g(x_0 e^{-\tau}) d\tau = - \int_1^e g(x_0 u) du = \int_{e^{-t}}^1 g(x_0 u) du.
$$
If the change of variable $v = e^{-t}u$ is used in the last integral, then $dv = e^{-t}du$ and
$$
e^{-t} \int_0^1 g(x_0 e^{-t}u) du = \int_0^{e^{-t}} g(x_0 v) dv = \int_0^{e^{-t}} g(x_0 u) du,
$$
where the dummy variable $v$ is changed back to $u$ in the last step. Putting these two facts together,
$$
y(t) + \int_0^1 g(x(t)u) du = e^t \left[ - \int_0^1 g(x_0 u) du + \int_0^{e^{-t}} g(x_0 e^{-\tau}) d\tau + e^{-t} \int_0^1 g(x_0 e^{-t}u) du \right]
$$
$$
= e^t \left[ - \int_0^1 g(x_0 u) du + \int_0^{e^{-t}} g(x_0 u) du + \int_0^{e^{-t}} g(x_0 u) du \right]
$$
$$
= e^t \left[ - \int_0^1 g(x_0 u) du + \int_0^1 g(x_0 u) du \right] = 0.
$$
Therefore, the set $W^s = \{(x, y) \mid y = - \int_0^1 g(xu) du\}$ is invariant. Moreover, if $t$ goes to infinity, $x(t) = x_0 e^{-t}$ goes to zero, and
$$
y(t) = - \int_0^1 g(x(t)u) du = - \int_0^1 g(x_0 e^{-t}u) du \to - \int_0^1 g(0) du = 0.
$$
Therefore, $W^s = \{(x, y) \mid y = - \int_0^1 g(xu) du\}$ is the stable manifold.