1. Since $V \geq 0$ and 
\[ \frac{1}{2} \dot{V} = (x + y)(\dot{x} + \dot{y}) + x^3 \dot{x} = (x + y)(-x^3) + x^3(y - 2x) = -3x^4 \leq 0, \]
by Lyapunov stability theorem, the origin is Lyapunov stable.

2. (a) $\frac{1}{2} \dot{V} = x \dot{x} + Ay \dot{y} = (3A - 1)x^2 y^2$.

(b) If $A = 1/3$, then $\dot{V} = 0$. Since $V \geq 0$ and $\dot{V} \leq 0$, then the origin is Lyapunov stable.

Remark: Although any $A \geq 1/3$ works, it is better to choose $A = 1/3$ because from $\dot{V}$, we know that $V(x(t)) = V(x_0)$ and the solution $x(t)$ stays on an ellipse—there is more information to tell than just by choosing some $A > 1/3$ (we only know that $V$ decreases). We can also easily understand the long term behaviour of this system, by studying the movement on the ellipses. For example, if the initial condition $(x_0, y_0)$ is in the first quadrant, then $x(t)^2 + y(t)^2/3 = x_0^2 + y_0^2/3$, and the solution $(x(t), y(t))$ converges to $(0, \sqrt{3x_0^2 + y_0^2})$.

3. (a) Since $(0, 0)$ is a fixed point of $(u, v)$-system, from the change of variable $u = x - x_0, v = y - y_0$, we know that $(x_0, y_0)$ must also be a fixed point of $(x, y)$-system (fixed points will be mapped to fixed points during any change of variables). Therefore,
\[ -x_0 + y_0 = 0, \quad x_0 - 2y_0 + 1 = 0 \]
or $x_0 = y_0 = 1$. The corresponding $(u, v)$-system is
\[ \dot{u} = -u + v, \quad \dot{v} = u - 2v. \]

(b) We are looking for a quadratic function $V(u, v) = u^2 + av^2 + buv$ for some constants $a > 0$ and $b$ such that $b^2 - 4a < 0$ (to make sure that the quadratic function $V$ is non-negative). Since
\[ \dot{V} = (2u + bv)\dot{u} + (2av + bu)\dot{v} = (2u + bv)(-u + v) + (2av + bu)(u - 2v) = (b - 2)u^2 + (b - 4a)v^2 + (2 - 3b + 2a)uv, \]
we have to choose the constant $a$ and $b$ such that $\dot{V} \leq 0$. Without loss of generality, we can let the coefficient of $uv$ in $\dot{V}$ vanish. That is,
\[ 2 - 3b + 2a = 0 \]

Together with the condition $b - 2 < 0$ and $b - 4a < 0$. Substituting $a = 3b/2 - 1$ into $\dot{V}$, we get
\[ \dot{V} = (b - 2)u^2 + (4 - 5b)v^2, \]

which requires $b \in (4/5, 2)$. On the other condition, we need to choose $a$ and $b$ such that $b^2 - 4a < 0$ (for $V$ to be non-negative), which is equivalent to
\[ b^2 - 4a = b^2 - 6b + 4 < 0 \]

In fact, for any $b \in (4/5, 2)$, the above inequality is satisfied identically (we only have to check the two end points of the interval), and we can choose $b = 1$ (and hence $a = 1/2$) for simplicity. Therefore, $V(u, v) = u^2 + v^2/2 + uv$. 

Answers to Exercise Sheet 3: Stability
(c) The Lyapunov function for \((x, y)\)-system can be chosen by transforming that for \((u, v)\)-system, that is
\[
\tilde{V}(x, y) = V(u, v) = V(x - x_0, y - y_0) = (x - 1)^2 + (y - 1)^2/2 + (x - 1)(y - 1).
\]
Then \(\dot{\tilde{V}}(x, y) = V(x, y) \geq 0\) and
\[
\dot{\tilde{V}}(x, y) = \dot{V}(u, v) \leq 0
\]
and \(\tilde{V} = 0\) if and only if at the fixed point \((x_0, y_0)\). As a result, the fixed point \((x_0, y_0) = (1, 1)\) is (asymptotically) stable.

4. The linearised system near the origin is
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
-\sigma & \sigma & 0 \\
r & -1 & 0 \\
0 & 0 & -b
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
\]
and the eigenvalues are the roots of
\[
\det\left(\begin{array}{ccc}
\lambda + \sigma & -\sigma & 0 \\
-r & \lambda + 1 & 0 \\
0 & 0 & \lambda + b
\end{array}\right) = \lambda + b \det\left(\begin{array}{ccc}
\lambda + \sigma & -\sigma & 0 \\
-r & \lambda + 1 & 0 \\
0 & 0 & \lambda + b
\end{array}\right) = (\lambda + b)(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)).
\]
That is
\[
\lambda_1 = -b < 0, \quad \lambda_{2,3} = \frac{-\sigma - 1 \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}}{2}.
\]
Since \(\lambda_2 + \lambda_3 = -\sigma - 1 < 0\), any complex roots must have negative real parts. If both roots \(\lambda_2\) and \(\lambda_3\) are real, then for \(r \in (0, 1)\), \(\lambda_2\lambda_3 = \sigma(1 - r) > 0\), which implies that both \(\lambda_2\) and \(\lambda_3\) have the same (negative) sign. That is, all three roots have negative real parts, and the origin stable node.

Using \(V = x^2 + \sigma y^2 + \sigma z^2\) we get
\[
\frac{1}{2} \dot{V} = \sigma x(y - x) + \sigma y(r x - y - xz) + \sigma z(-bz + xy).
\]
After a little manipulation
\[
\frac{1}{2\sigma} \dot{V} = -\left(x + \frac{(r + 1)}{2} y\right)^2 - \left(1 - \frac{(r + 1)^2}{4}\right) y^2 - bz^2
\]
which is strictly less than zero if \((x, y, z) \neq (0, 0, 0)\) provided \((r + 1)^2 < 4\), i.e. \(0 < r < 1\) (as \(r > 0\) by assumption).

*Remark:* In the lecture note, the choice \(V = rx^2 + \sigma y^2 + \sigma z^2\) was made, showing the same conclusion. In fact, there are many other choices of the coefficients of \(x^2\) leading to the same result.