MATH 44041/64041 Applied Dynamical Systems

Answers to Exercise Sheet 1 : Phase portrait, fixed points and invariant sets

1. We are given
   \[ \dot{x} = x^2 + (2 - \mu)x - 2\mu = (x - \mu)(x + 2) \]
   so the sketch of the right hand side depends on whether \( \mu < -2 \), \( \mu = -2 \) or \( \mu > -2 \) (see Figure 3).

   ![Figure 1](image.png)
   Figure 1: \( \dot{x} \) against \( x \) showing the direction of flow: left diagram \( \mu < -2 \); middle diagram \( \mu = -2 \); right diagram \( \mu > -2 \).

   (a) Case \( \mu < -2 \): solutions with initial condition \( x_0 < -2 \) tend to \( \mu \) as \( t \to \infty \) and if \( x_0 > -2 \) then solutions tend to infinity.

   (b) Case \( \mu = -2 \): solutions with initial condition \( x_0 < -2(= \mu) \) tend to \( -2 \) as \( t \to \infty \) and if \( x_0 > -2 \) then solutions tend to infinity.

   (c) Case \( \mu > -2 \): solutions with initial condition \( x_0 < \mu \) tend to \( -2 \) as \( t \to \infty \) and if \( x_0 > \mu \) then solutions tend to infinity.

   **Remark:** We will study the change of the behaviour of the system depending on parameters later in this module, something called bifurcation.

2. Since
   \[ \dot{x} = \mu x(1 - x) \]
   we can separate variables so
   \[ \int_{x_0}^{x} \frac{dx}{x(1-x)} = \mu \int_{0}^{t} dt = \mu t. \]
   Using partial fractions \( \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x} \) so doing the integral with \( x, x_0 > 0 \) gives
   \[ \log \frac{x}{x_0} - \log \left| \frac{1-x}{1-x_0} \right| = \log \left( \frac{|1-x_0|}{x_0} \cdot \frac{x}{|1-x|} \right) = \mu t \]
   which, after a little manipulation (especially the fact that \( |1-x(t)|/|1-x_0| = (1-x(t))/(1-x_0) \), implies
   \[ x = \frac{x_0}{x_0 + (1-x_0)e^{-\mu t}} \]
Figure 2: $\dot{x}$ against $x$ showing the direction of flow.

where we have chosen this form to make it obvious that if $x_0 > 0$ then $x \to 1$ as $t \to \infty$.

This is obvious from the sketch of the right hand side of the $\dot{x}$ equation: if $x \in (0,1)$ then $\dot{x} > 0$ as so $x$ increases, whilst if $x_0 > 1$ then $\dot{x} < 0$ and so $x$ decreases. At $x = 1$, $\dot{x} = 0$ so this is a stationary point of the equation (see Figure 2).

3. Suppose $\phi(x_0, t) = x_0 e^{at}$, $a \neq 0$. Then

$$\phi(\phi(x_0, t), s) = (x_0 e^{at}) e^{as} = x_0 e^{a(t+s)} = \phi(x_0, t+s).$$

The semi-group property is satisfied.

Suppose $\phi(x_0, t) = x_0 \sin(\omega t)$, $\omega \neq 0$. Then

$$\phi(\phi(x_0, t), s) = (x_0 \sin(\omega t)) \sin \omega s \neq x_0 \sin(\omega(t+s))$$

so this does not satisfy the property.

Similarly if $\phi(x_0, t) = x_0 + at$, $a \neq 0$. Then

$$\phi(\phi(x_0, t), s) = (x_0 + at) + as = x_0 + a(t+s) = \phi(x_0, t+s),$$

and so the semigroup property does hold.

Whilst if $\phi(x_0, t) = x_0 t$ then

$$\phi(\phi(x_0, t), s) = (x_0 t)s = x_0(ts)$$

and

$$\phi(x_0, t+s) = x_0(t+s)$$

so the semigroup property does not hold.

4. By definition,

$$\varphi(\varphi(x_0, t), s) = \frac{\varphi(x_0, t)}{\sqrt{1 + 2s\varphi(x_0, t)^2}} = \frac{x_0}{\sqrt{1 + 2s \frac{x_0^2}{1+2x_0^2}}} = \frac{x_0}{\sqrt{1 + 2tx_0^2 + 2sx_0^2}} = \frac{x_0}{\sqrt{1 + 2(t+s)x_0^2}} = \varphi(x_0, t+s).$$

If such a function $f(x)$ exists such that $\varphi(x_0, t)$ is a solution to $\dot{x} = f(x)$, then we must have $\frac{d}{dt} \varphi(x_0, t) = f(\varphi(x_0, t))$ for any $t > 0$. From the derivative

$$\frac{d}{dt} \varphi(x_0, t) = x_0\left(-\frac{1}{2}\right) (1 + 2tx_0^2)^{-3/2} 2x_0^2 = -x_0^3 (1 + 2tx_0^2)^{-3/2} = -\left(\frac{x_0}{\sqrt{1 + 2tx_0^2}}\right)^3,$$
we get \( f(x) = -x^3 \) by inspection. Remark: because \( \varphi(x_0, t) \) satisfies the semi-group property, the law of the dynamics characterised by \( f \) can also be determined from the relation \( \frac{d}{dt}\varphi(x_0, t) = f(\varphi(x_0, t)) \) evaluated at time \( t = 0 \), that is,

\[
f(x_0) = f(\varphi(x_0, t)) \bigg|_{t=0} = \frac{d}{dt}\varphi(x_0, t) \bigg|_{t=0} = -x_0^3 \left(1 + 2tx_0^2\right)^{-3/2} \bigg|_{t=0} = -x_0^3.
\]

5. From the equations

\[
\dot{x} = A - x - xy, \quad \dot{y} = -y + xy,
\]
if we take derivative of \( G(x, y) = x + y - A \) along the line defined by \( G(x, y) = 0 \), then

\[
\dot{G}|_{G=0} = \dot{x} + \dot{y} = (A - x - y)|_{x+y-A=0} = 0.
\]

Therefore, the line \( x + y - A = 0 \) is invariant.

6. (Method based on conserved quantity) The conserved quantity can be obtained from the solution to

\[
\frac{dB}{dA} = \frac{\dot{B}}{\dot{A}} = \frac{\alpha A}{\beta B}.
\]

This separable ODE can be written as \( \beta BdA = \alpha AdA \) and then integrated to give \( Q(A, B) = \alpha A^2 - \beta B^2 = c \) for some constant. Here \( Q(A, B) \) is exactly the conserved quantity (which satisfies \( \frac{d}{dt}Q \equiv 0 \)). Then \( B \) loses the battle means that \( A(T) > 0 \) at the time \( T \) when \( B(T) = 0 \). In other words, \( Q(A, B) = Q(A(T), B(T)) = \alpha A(T)^2 > 0 \). Substituting the initial condition into it, we get the condition

\[
Q(A(0), B(0)) = \alpha A_0^2 - \beta B_0^2 > 0.
\]

(Method based on explicit solutions) The system \( \dot{A} = -\beta B, \dot{B} = -\alpha A \), or equivalently \( \frac{d}{dt} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \) is linear, and the solution can be obtained either by matrix exponential based the fact that

\[
\begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix}^{2n} = (\alpha \beta)^n I, \quad \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix}^{2n+1} = (\alpha \beta)^n \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix}.
\]

Alternatively this system can be reduced to a second order ODE. That is, \( \ddot{A} = \frac{d}{dt} \dot{A} = \frac{d}{dt}(-\beta B) = -\beta \dot{B} = \alpha \beta A \). Since \( \alpha \) and \( \beta \) are positive constants, the solution \( A \) is given by

\[
A(t) = a \cosh(\sqrt{\alpha \beta} t) + b \sinh(\sqrt{\alpha \beta} t)
\]

for some constants \( a \) and \( b \). Then from \( \dot{A} = -\beta B \), we get

\[
B(t) = -\frac{1}{\alpha} \dot{A}(t) = -\sqrt{\frac{\beta}{\alpha}} \left( a \sinh(\sqrt{\alpha \beta} t) + b \cosh(\sqrt{\alpha \beta} t) \right).
\]

The initial condition \( A(0) = A_0, B(0) = B_0 \) becomes

\[
A_0 = A(0) = a, \quad B_0 = -\sqrt{\frac{\alpha}{\beta}} b,
\]

or \( a = A_0, b = -B_0\sqrt{\beta/\alpha} \). When \( B(t) \) is zero at time \( T \), then

\[
0 = B(T) = -\sqrt{\frac{\beta}{\alpha}} \left( a \sinh(\sqrt{\alpha \beta} T) + b \cosh(\sqrt{\alpha \beta} T) \right),
\]
or \( \tanh(\sqrt{\alpha \beta T}) = -b/a = \sqrt{\frac{\beta B_0}{\alpha A_0}}. \) Then \( B \) loses the battle implies that \( A(T) > 0, \) which is
\[
A(T) = a \cosh(\sqrt{\alpha \beta T}) + b \sinh(\sqrt{\alpha \beta T}) = \cosh(\sqrt{\alpha \beta T}) \left[ a + b \tanh(\sqrt{\alpha \beta T}) \right]
= \cosh(\sqrt{\alpha \beta T}) \left( A_0 - \frac{\beta B_0^2}{\alpha A_0} \right) > 0,
\]
which is \( \alpha A_0^2 > \beta B_0^2. \)

**Remark:** As we can see, a conserved quantity, if it exists, can be used to solve qualitative problems much more effective.

7. Set \( G(x, y) = y - x \) and note that the line is \( G(x, y) = 0. \) Thus
\[
\dot{G} = \dot{y} - \dot{x} = -2x + 2y + 4x(x - y).
\]
Hence if \( (x, y) \) is on the line \( \{(x, y) \mid G = 0\} \) (i.e. \( y = x \)), then \( \dot{G} = -2x + 2y + 4x(x - y)\big|_{y=x} = 0. \)
Therefore \( G = 0, \) and the line \( y = x \) is invariant.

8. Set \( G(x, y) = x^2 + y^2 - 1, \) so
\[
\dot{G} = 2x \dot{x} + 2y \dot{y} = 2\left( -(x^2 + y^2) + (x^2 + y^2)^2 \right) = 2(x^2 + y^2)(x^2 + y^2 - 1).
\]
Hence \( \dot{G} = 2G(G + 1) = 0 \) and if \( (x, y) \) is on the circle \( \{(x, y) \mid G = 0\}. \) Therefore, the unit circle \( \{(x, y) \mid x^2 + y^2 - 1 = 0\} \) is invariant.

9. [Be a little careful about the notation, probably more clear to write the original ODE as
\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.
\]
] The line \( y = mx \) can be written as the set
\[
S = \{(x, y) \mid G(x, y) = 0\}, \quad \text{with } G(x, y) = mx - y = (m -1) \begin{pmatrix} x \\ y \end{pmatrix}.
\]
Then
\[
\frac{d}{dt} G(x, y) = (m -1) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (m -1) A \begin{pmatrix} x \\ y \end{pmatrix}.
\]
On the line \( y = mx, \) the rate of change of the function \( G(x, y) \) is
\[
\frac{d}{dt} G(x, y)\bigg|_{G(x,y)=0} = (m -1) A \begin{pmatrix} x \\ y \end{pmatrix}\bigg|_{y=mx} = (m -1) A \begin{pmatrix} x_{mx} \\ mx \end{pmatrix} = x (m -1) A \begin{pmatrix} 1 \\ m \end{pmatrix}.
\]
Since \( \begin{pmatrix} 1 \\ m \end{pmatrix} \) is an eigenvector of \( A \) corresponding to some eigenvalue \( \lambda, \) we have
\[
\frac{d}{dt} G(x, y)\bigg|_{G(x,y)=0} = x (m -1) A \begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda x (m -1) \begin{pmatrix} 1 \\ m \end{pmatrix} = 0.
\]
Therefore, the line \( y = mx \) is invariant.

(Alternative approach) Instead of the vector version, we can also proceed by assuming the coefficients of \( A \) and working with each component of the equation. Assume \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \) then the original ODE is equivalent to
\[
\dot{x} = ax + by, \quad \dot{y} = cx + dy.
\]
Then from the fact that $\mathbf{e} = (1, m)^t$ is the eigenvector associated with the eigenvalue $\lambda$, we get
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix},
\]
or
\[a + bm = \lambda, c + dm = m\lambda.\]
Eliminating $\lambda$, we get
\[c + dm = m(a + bm). \tag{1}\]

On the other hand,
\[
\frac{d}{dt}G(x, y)|_{y = mx} = m\dot{x} - \dot{y} = m(ax + by) - (cx + dy)|_{y = mx} = (m(a + m) - c - dm)x = 0,
\]
because of (1).

10. We can show the line is invariant using the definition (if the initial condition $x_0$ on the line, then the solution $x(t)$ is also on the line). If the initial condition $x_0$ is on the line through the origin parallel to the eigenvector $\mathbf{e}$ (associated with the eigenvalue $\lambda$), then $x_0 = \mu\mathbf{e}$ for some constant $\mu$. The fact that $\mathbf{e}$ is a simple real eigenvalue implies that the solution to the system $\dot{x} = Ax$ is
\[x(t) = \mu e^{\lambda t} \mathbf{e} = e^{\lambda t}x_0,
\]
which means that $x(t)$ is also on the line through the origin parallel to $\mathbf{e}$.

NB. If $A$ has $n$ eigenvalues $\{\mathbf{e}_i\}$ (with eigenvalues $\{\lambda_i\}$ forming a basis of $\mathbb{R}^n$, then for any vector $x_0$ that is expressed in this basis as
\[x_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n
\]
for some constant coefficients $\{c_i\}$, then the solution to $\dot{x} = Ax, x(0) = x_0$ is
\[x(t) = e^{At}x_0 = c_1e^\lambda e^{\lambda_1 t}\mathbf{e}_1 + c_2e^\lambda e^{\lambda_2 t}\mathbf{e}_2 + \cdots + c_ne^\lambda e^{\lambda_n t}\mathbf{e}_n.
\]

11. (a) If
\[A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}
\]
then the eigenvalues satisfy $(s - 4)(s - 1) + 2 = 0$, i.e. $s^2 - 5s + 6 = 0$ with eigenvalues $s_1 = 2$ and $s_2 = 3$. The eigenvector equations are
\[
\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = 2 \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} = 3 \begin{pmatrix} 1 \\ b \end{pmatrix}
\]
so $a = 1$ and $b = \frac{1}{2}$, so take the eigenvectors as $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Thus we take $U^{-1}$ to be the matrix with columns the eigenvectors:
\[U^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \text{or} \quad U = -\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}.
\]
In the new coordinates $y = Ux$ the equation is $\dot{y} = \Lambda y$ with
\[
\Lambda = UAU^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.
\]
(b) If

\[ A = \begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix} \]

the eigenvalue equation is \((s + 4)(s - 2) + 10 = 0\) or \(s^2 + 2s + 2 = 0\) i.e. \(s = -1 \pm i\). The eigenvector corresponding to the eigenvalue \(-1 + i\) is \(\begin{pmatrix} 2 \\ -3 - i \end{pmatrix}\) so we take the real and imaginary parts as the columns of the transformation matrix:

\[ U^{-1} = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix} \quad \text{or} \quad U = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} \]

In the new coordinates \(y = Ux\) the equation is \(\dot{y} = \Lambda y\) with

\[ \Lambda = UAU^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \]

giving a stable focus \((\dot{r} = -r, \dot{\theta} = -1)\).

![Figure 3: Phase portraits for Problem 11 in new coordinates (left) and original coordinates (right). Notice that the direction of the notation in (b) is different in the original and the transformed plane (will be the same if the two columns in the transformation matrix is switched).](image)

12. The eigenvalue equation is \(s^2 - ts + d = 0\) where \(t = \text{tr}(A)\) and \(d = \det(A)\) are the trace and determinant of the matrix, hence the eigenvalues are \(s_{\pm} = \frac{1}{2} \left( t \pm \sqrt{t^2 - 4d} \right) \).

   (a) If \(t^2 - 4d < 0\) then the eigenvalues are complex conjugate and the real part is negative iff \(t < 0\).

   (b) If \(t^2 - 4d \geq 0\) then the eigenvalues are real and they are both negative iff the larger is negative, i.e. if \(t + \sqrt{t^2 - 4d} < 0\) or \(t < 0\) and \(d > 0\).

Putting all these together, the necessary and sufficient condition is \(t < 0\) and \(d > 0\) (see Figure 2). These conditions imply that solutions of the linear equation tend to zero as \(t \to \infty\).
Figure 4: The regions of trace $t$ and determinant $d$ for Problem 12.

13. For the ODE $\dot{x} = 1 - xt, x(0) = 0$, the Picard’s iteration for the sequence $\{x^{(n)}(t)\}$ of approximation is given by

$$x^{(n+1)}(t) = x_0 + \int_0^t (1 - x^{(n)}(s)s)\,ds = \int_0^t (1 - x^{(n)}(s)s)\,ds, \quad x^{(0)}(t) = 0.$$ 

Therefore,

$$x^{(1)}(t) = \int_0^t (1 - sx^{(0)}(s))\,ds = s,$$

$$x^{(2)}(t) = \int_0^t (1 - sx^{(1)}(s))\,ds = \int_0^t (1 - s^2)\,ds = s - \frac{1}{3}s^3.$$