6 Maps and their bifurcation

6.1 Fixed points and periodic orbits of maps

Recall that a discrete time system (or map) is defined by a difference equation

$$x_{n+1} = f_\mu(x_n), \qquad x_n \in \mathbb{R}^n,$$

with parameter(s) μ (the subscript μ may be omitted if no confusion arises). Similar to continuous dynamical system, simple solutions include:

i) **Fixed Points:** $x_{n+1} = x_n$, that is solutions of $x^* = f(x^*)$.

ii) **Periodic orbits:** $(x_0, \ldots, x_{p-1})$ with $x_k = f(x_{k-1}), k = 1, \ldots, p-1$ and $x_0 = f(x_{p-1})$. Therefore,

$$x_k = f^p(x_k) = \underbrace{f(\cdots(f(x_k))\cdots)}_{p \text{ iterations}}, \qquad k = 0, 1, 2, \cdots, p-1.$$

That is, periodic points are *fixed points* of an iterate f^p of the map. We usually work with smallest period.

The *stability* of fixed points or periodic orbits can also be studied via linearisation. If x_n is close to the fixed point, let $y_n = x_n - x^*$, then

$$y_n = x_n - x^* = f(x_{n-1}) - f(x^*) \approx (x_{n-1} - x^*)f'(x^*) = f'(x^*)y_{n-1},$$

and $y_n \approx [f'(x^*)]^n y_0$. Therefore, a fixed point x^* is *linearly stable* if $|f'(x^*)| < 1$.

For a periodic orbit with period p, the condition for the fixed points is

$$|(f^p)'(x_k)| < 1 \qquad k = 0, 1, 2, \cdots p-1.$$

If fact, we only need to check one k, since $(f^p)'(x_0) = (f^p)'(x_1) = \cdots = (f^p)'(x_{p-1})$. Using the chain rule (try the case of $p = 2$ and $p = 3$ to see how it works),

$$\begin{aligned}
\frac{d}{dx} f^p(x_k) &= f'(\underbrace{f(\cdots f(x_k)\cdots)}_{p-1 \text{ iterations}})f'(\underbrace{f(\cdots f(x_k)\cdots)}_{p-2 \text{ iterations}})\cdots f'(x_k) \\
&= f'(x_{k+p-1})f'(x_{k+p-2})\cdots f'(x_k) \\
&= f'(x_{k-1})f'(x_{k-2})\cdots f'(x_k) \\
&= f'(x_0)f'(x_1)\cdots f'(x_{p-1}). \qquad (6.1)
\end{aligned}$$

So there is only one condition for the linear stability of a periodic orbit: $\prod_{k=0}^{p-1} |f'(x_k)| < 1$.

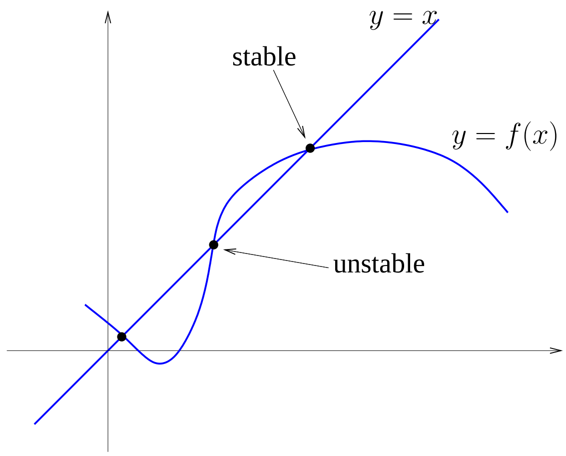
The concept of *invariant set* can also be defined for maps, but is less used than that for the continuous dynamical systems.

Example 6.1. Let λ be any non-zero constant. Then the parabola $P = \{(x, y) \mid y = x^2\}$ is an invariant set for the map $x_{n+1} = \lambda x_n, \quad y_{n+1} = \lambda^2 y_n$. In fact, if $(x_n, y_n) \in P$, then $y_n = x_n^2$ and

$$y_{n+1} = \lambda^2 y_n = \lambda^2 x_n^2 = (\lambda x_n)^2 = x_{n+1}^2.$$

That is $(x_{n+1}, y_{n+1}) \in P$ as well. Therefore, the parabola P is invariant.

Because of the points x_n are discrete in space, special graphic tools (other than the phase portrait) are used. First, fixed points for the one dimensional map $x_{n+1} = f(x_n)$ can be viewed as the intersection of the straight line $y = x$ and the curve $y = f(x)$. If $f'(x^*)$ is positive at the fixed point x^*, the stability can also be determined graphically, by comparing $f'(x^*)$ with the slope of $y = x$ (see Figure 6.1).



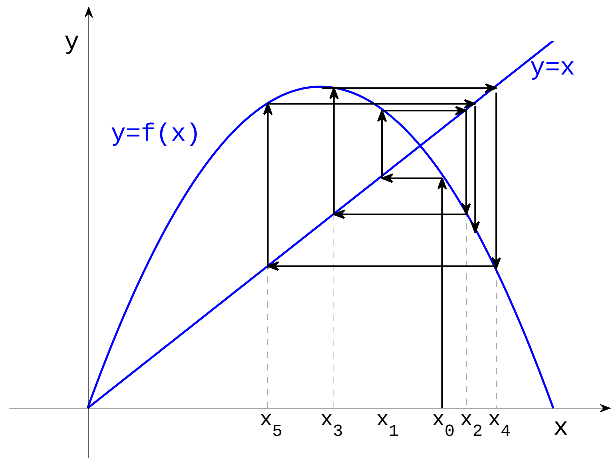
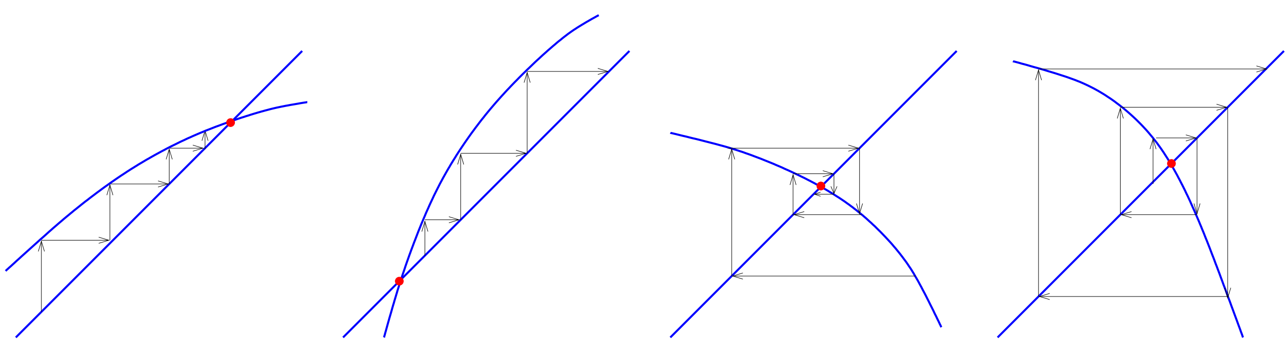


Figure 6.1: Left figure: graphic representation of the fixed points at the intersection between the straight line $y = x$ and the curve $y = f(x)$. Right figure: cobweb diagram showing the iteration of the map $x_{n+1} = f(x_n)$.

The iteration of the trajectory $x_0, x_1, \cdots$ can be viewed from the cobweb diagram (right figure in Figure 6.1): (1) the vertical line $x = x_n$ intersect the curve $y = f(x)$ at $(x_n, f(x_n)) = (x_n, x_{n+1})$; (2) the horizontal line through (x_n, x_{n+1}) intersect the line $y = x$ at $(x_{n+1}, f(x_{n+1})$; (3) then the vertical line through (x_{n+1}, x_{n+1}) becomes $x = x_{n+1}$ and the whole process can be continued again.



$0 < f'_\mu(x^*) < 1$ $\quad$ $f'_\mu(x^*) > 1$ $\quad$ $-1 < f'_\mu(x^*) < 0$ $\quad$ $f'_\mu(x^*) < -1$

Figure 6.2: The behaviour near a fixed point in terms of the cobweb diagram

The behaviour of the map $x_{n+1} = f_\mu(x_n)$ near a fixed point can be understood using the cobweb diagram as shown in Figure 6.2. While the stability (inward towards the fixed point x^* or not) is determined by whether $|f'_\mu(x^*)|$ is greater than unit, the sign of $f'_\mu(x^*)$ determines whether the diagram looks like stairs ($f'_\mu(x^*) > 0$) or spirals ($f'_\mu(x^*) < 0$).

6.2 Bifurcation of maps

Similarly, as the parameter μ in the map $x_{n+1} = f_\mu(x_n)$ varies, bifurcation could occur at the fixed point $x^* = f_\mu(x^*)$ if $|f'_\mu(x^*)|$ passes one. To compare with the continuous dynamical systems $\dot{x} = f_\mu(x)$, we can consider the analogous discrete maps $x_{n+1} = x_n + f_\mu(x_n)$ (instead of $x_{n+1} = f_\mu(x_n)$, such that the fixed points in both cases coincide, that is $f_\mu(x^*) = 0$ and the bifurcation diagrams are exactly the same.

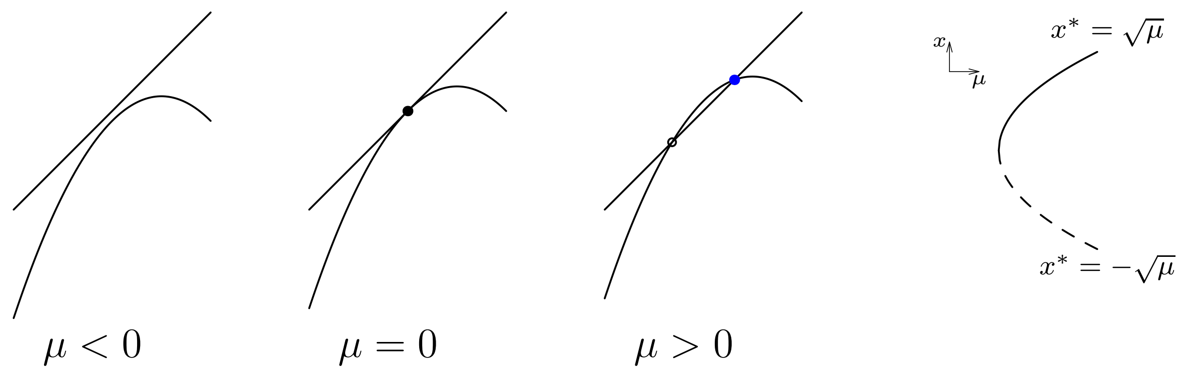


Figure 6.3: Saddle-node (tangential) bifurcation.

Saddle-node (tangential) bifurcation for $x_{n+1} = \mu + x_n - x_n^2$: If $\mu > 0$, there are two fixed points $x^*_\pm = \pm\mu^{1/2}$ (two intersection points between the curve $y = x$ and $y = \mu + x - x^2$); the fixed point $x^*_+ = \mu^{1/2}$ is stable but $x^*_- = -\mu^{1/2}$ is not stable. If $\mu < 0$, there is no fixed point. Because bifurcation occurs when the straight line $y = x$ touches the parabola $y = \mu + x - x^2$ tangentially at $\mu = 0$, this saddle-node bifurcation is also called tangential bifurcation (see Figure 6.3).

Transcritical bifurcation for $x_{n+1} = (1 + \mu)x_n - x_n^2$: There are always two fixed points $x^* = 0$ and $x^* = \mu$. The fixed point $x^* = 0$ is stable for $\mu < 0$, but becomes unstable for $\mu > 0$, while the other fixed point $x^* = \mu$ is stable.

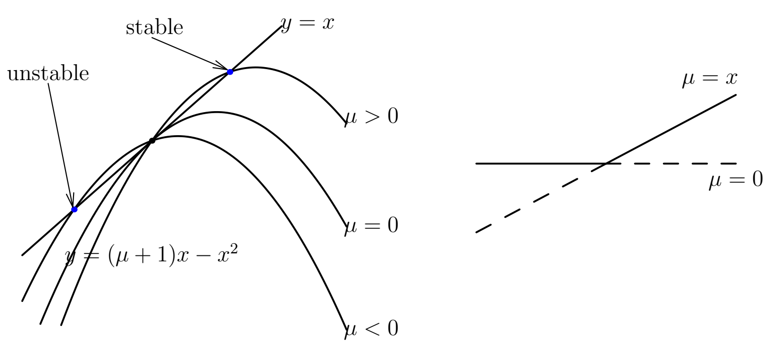


Figure 6.4: Transcritical bifurcation.

Supercritical pitchfork bifurcation for $x_{n+1} = (1 + \mu)x_n - x_n^3$: When $\mu < 0$, there is only one fixed point $x^* = 0$, which is stable. When $\mu > 0$, there are three fixed points; $x^* = \pm\mu^{1/2}$ are stable, but $x^* = 0$ unstable.

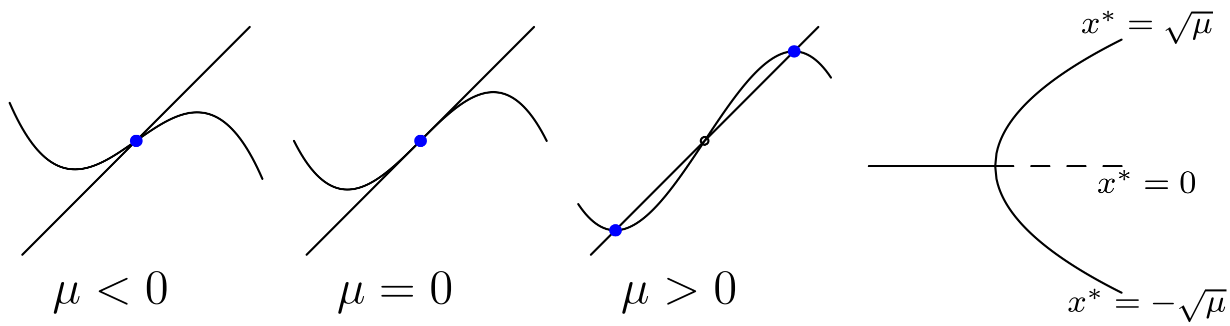


Figure 6.5: Pitchfork bifurcation.

Remark. Although the bifurcation diagrams of the three bifurcations look the same as those for the continuous differential equations, the situation for discrete maps is more complicated:

(a) The bifurcation are valid only *locally* for discrete maps. Take the map $x_{n+1} = 1 + x_n - x_n^2$ ($\mu = 1$) and the initial condition $x_0 = 10$ (or any initial such that x_0 is large), then $x_n \to -\infty$ as n goes to infinity, different from the stable fixed point $x^* = \mu^{1/2} = 1$. But for the continuous dynamical system $\dot{x} = 1 - x^2$, $x(t)$ always converges to $x^* = \mu^{1/2} = 1$, if x_0 is large.

(b) Another bifurcation could happen along the stable fixed point, as μ further increases or decreases. The Jacobian of the map $x_{n+1} = \mu + x_n - x^2$ at the stable fixed point $x^* = \mu^{1/2}$ is

$$f'_\mu(x^*) = 1 - 2x^* = 1 - 2\mu^{1/2}.$$

Then as $\mu > 1$, $f'_\mu(x^*) < -1$, which becomes unstable (a period-doubling bifurcation happens as we show below).

For continuous dynamical systems, Hopf bifurcation is common when stable spiral becomes unstable spiral (real parts of the eigenvalue becomes positive), and periodic solution appears. For maps, the analogous situation is period-two bifurcation: the original fixed point becomes unstable, and a period-two orbit appear. This will be examined in the next subsection, for the special logistic map.

6.3 Logistic map

The *logistic map* is the simplest quadratic family of maps

$$f_\mu(x) = \mu x(1 - x), \qquad \mu \geq 0,$$

in which chaotic behaviours can arise. In the context of population dynamics, the two terms μx and $-\mu x^2$ in this map can be interpreted as reproduction and starvation (density-dependent mortality) respectively.

If this map is invariant on the interval $[0, 1]$, then $\mu \in [0, 4]$, since we only have to make sure $\max_{x\in[0,1]} f_\mu(x) = f_\mu(1/2) = \mu/4 \leq 1$. The behaviour of the map for small and moderately large μ can be explained by examining the stability of the fixed points and the periodic orbits.

Fixed Points: $x^* = \mu x^*(1 - x^*)$. So $x^* = 0$ or $x^* = (\mu - 1)/\mu$ provided $\mu \geq 1$.

Linear Stability: First $f'_\mu(x) = \mu - 2x\mu$. If $0 \leq \mu < 1$, the fixed point $x^* = 0$ is stable since $|f'_\mu(0)| = \mu < 1$, and the fixed point $x^* = (\mu - 1)/\mu$ is not in the range $[0, 1]$. As $\mu \geq 1$, the fixed point $x^* = 0$ becomes unstable. But $x^* = (\mu - 1)/\mu \in (0, 1)$ become stable, as long as

$$|f'_\mu\big((\mu - 1)/\mu\big)| = |2 - \mu| < 1,$$

or $1 < \mu < 3$. Because the fixed points $x^* = 0$ and $x^* = (\mu - 1)/\mu$ exchange stability at $\mu = 1$, this is a transcritical bifurcation.

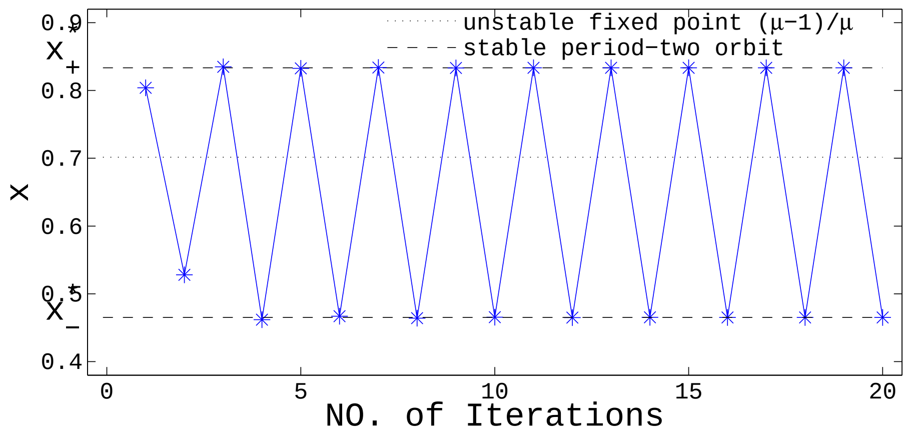


Figure 6.6: The fixed point $x^* = (\mu - 1)/\mu$ becomes unstable as $\mu > 3$, and a period-two orbit emerges (the iteration for $\mu = 3.35$ is plotted here).

Period-doubling bifurcation: As μ passes 3, $f'_\mu\big((\mu - 1)/\mu\big)$ passes -1 and $x^* = (\mu - 1)/\mu$ becomes unstable (see Figure 6.6 for sample iterations at $\mu = 3.35$). A period-two orbit (x^*_+, x^*_-) appears, such that

$$x^*_+ = f_\mu(x^*_-), \quad x^*_- = f_\mu(x^*_+).$$

In other words, both x^*_+ and x^*_- are fixed points of $x = f_\mu(f_\mu(x))$, but not fixed points of $x = f_\mu(x)$. This is called *period-doubling bifurcation*, signified by $f_{\mu^*}(x^*) = -1$ at $\mu^* = 3$.

Since

$$x - f_\mu\big(f_\mu(x)\big) = x(\mu x - \mu + 1)(\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1),$$

all fixed points of $x = f_\mu\big(f_\mu(x)\big)$ are

$$x^* = 0, \quad x^* = \frac{\mu - 1}{\mu}, \quad x^*_\pm = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}.$$

The first two are inherited from $x^* = f_\mu(x^*)$, and the last two form the period two orbits, solving the quadratic equation $\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1 = 0$. A more involved calculation shows that the this period-two orbit loses its stability, when the modulus

$$\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\Big|_{x^*_\pm} = f'_\mu\big(f_\mu(x)\big) f'_\mu(x)\Big|_{x^*_\pm} = f'_\mu(x^*_+) f'_\mu(x^*_-)$$

is greater than unit. First from

$$x_+^* + x_-^* = \frac{\mu+1}{\mu}, \quad x_+^* x_-^* = \frac{\mu+1}{\mu^2},$$

the Jacobian $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x_\pm^*}$ can be simplified as

$$f'_\mu(x_+^*) f'_\mu(x_-^*) = \mu^2 (1-2x_-^*)(1-2x_+^*) = \mu^2\big(1 - 2(x_+^* + x_-^*) + 4x_-^* x_+^*\big) = 4 + 2\mu - \mu^2.$$

By solving $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x_\pm^*} = \pm 1$, we get $\mu = -1$ or $\mu = 3$ for $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x_\pm^*} = 1$ and $\mu = 1 \pm \sqrt{6}$ for $\frac{d}{dx} f_\mu\big(f_\mu(x)\big)\big|_{x_\pm^*} = -1$. We do not need to check the negative values of $\mu = -1$ or $\mu = 1 - \sqrt{6}$; in fact the fixed points $x_\pm^*$ exists only for $\mu \geq 3$. Therefore, the only possible bifurcation is at $\mu^* = 1 + \sqrt{6} \approx 3.449$, with $\frac{d}{dx} f_{\mu^*}\big(f_{\mu^*}(x)\big)\big|_{x_\pm^*} = -1$. The value -1 suggests another period-doubling bifurcation for $x = f_\mu(f_\mu(x))$, leading to period-four orbits, which are fixed points of $x = f_\mu(f_\mu(f_\mu(f_\mu(x))))$.

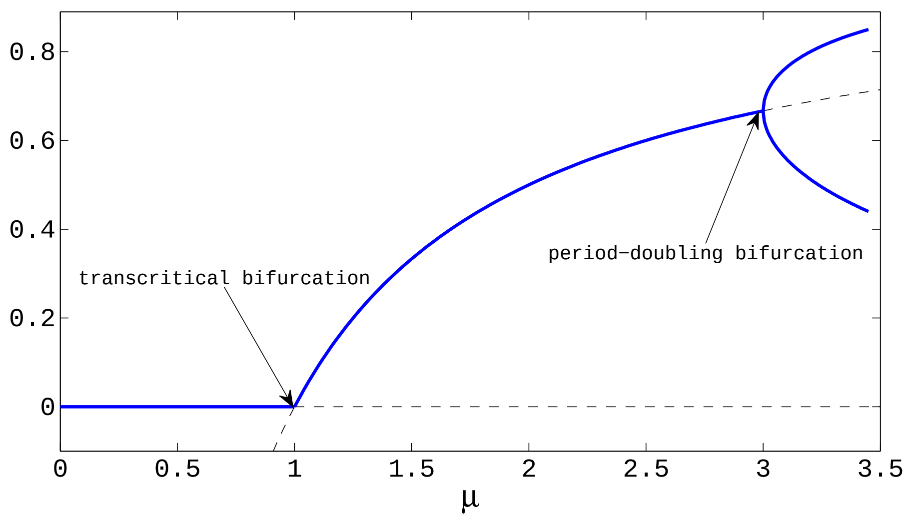


Figure 6.7: Bifurcation diagram for the logistic map, when μ is not too close to 4.

In fact there is an infinite cascade of 'period-doubling' bifurcations:

$$\begin{array}{ll} \mu_1 = 3 & \text{period } 1 \to 2 \\ \mu_2 = 1 + \sqrt{6} & \text{period } 2 \to 4 \\ \quad\vdots & \\ \mu_n & \text{period } 2^{n-1} \to 2^n. \end{array}$$

Moreover, μ_n has a finite limit (about 3.56995), when the period-doubling cascade ends and chaotic behaviours start. There is also a universal *Feigenbaum constant* defined as the limit of ratio between the lengths of two successive bifurcation intervals, i.e,

$$\lim_{n\to\infty} \frac{\mu_{n-1} - \mu_n}{\mu - \mu_{n+1}} \approx 4.669$$

Many other maps exhibit similar (period-doubling) bifurcations, and the above limiting ratio is independent of the details of the map.

Remark. Although the bifurcation diagrams for both pitchfork and period-doubling bifurcation look similar, the behaviours of the fixed points are totally different: new "fixed points" for period-doubling bifurcation are actually fixed points of $x = f_\mu(f_\mu(x))$ instead of $x = f_\mu(x)$.

6.4 Bifurcation of two-dimensional maps

The same approach can be used to study the bifurcation of two-dimensional maps, by looking at when the change in the parameter leads to eigenvalues of the *Jacobian matrix* with unit modulus.

Example 6.2. Consider the map

$$x_{n+1} = \mu y_n + x_n - x_n^2, \qquad y_{n+1} = x_n,$$

where $|\mu|$ is small. The fixed point (x^*, y^*) satisfies the equations $x = \mu x + x - x^2, y = x$. That is, there are two fixed points

$$(x_1^*, y_1^*) = (0,0), \qquad (x_2^*, y_2^*) = (\mu, \mu).$$

From the Jacobian matrix

$$J(x,y) = \begin{pmatrix} \frac{\partial}{\partial x}(\mu y + x - x^2) & \frac{\partial}{\partial y}(\mu y + x - x^2) \\ \frac{\partial}{\partial x} x & \frac{\partial}{\partial y} x \end{pmatrix} = \begin{pmatrix} 1-2x & \mu \\ 1 & 0 \end{pmatrix},$$

we get

$$J(x_1^*, y_1^*) = \begin{pmatrix} 1 & \mu \\ 1 & 0 \end{pmatrix}, \qquad J(x_2^*, y_2^*) = \begin{pmatrix} 1-2\mu & \mu \\ 1 & 0 \end{pmatrix}.$$

At the fixed point (x_1^*, x_2^*), the two eigenvalues are governed by

$$\det\big(\lambda I - J(x_1^*, y_1^*)\big) = \det\begin{pmatrix} \lambda - 1 & -\mu \\ -1 & \lambda \end{pmatrix} = \lambda^2 - \lambda - \mu = 0.$$

That is

$$\lambda_1^\pm = \frac{1 \pm \sqrt{1+4\mu}}{2}.$$

As μ passes zero, λ_1^+ pass 1 and this fixed point (x_1^*, y_1^*) becomes unstable.

At the fixed point (x_2^*, y_2^*), the two eigenvalues are governed by

$$\det\big(\lambda I - J(x_2^*, y_2^*)\big) = \det\begin{pmatrix} \lambda - 1 + 2\mu & -\mu \\ -1 & \lambda \end{pmatrix} = \lambda^2 - (1-2\mu)\lambda - \mu = 0.$$

That is,

$$\lambda_2^\pm = \frac{1-2\mu \pm \sqrt{(1-2\mu)^2 + 4\mu}}{2} = \frac{1-2\mu \pm \sqrt{1+4\mu^2}}{2}.$$

In this case, λ_2^- is close to zero ($|\lambda_2^+|$ is far away from unit) and can not trigger any instability. If μ is small and negative,

$$\sqrt{1+4\mu^2} > \sqrt{1+4\mu+4\mu^2} = 1+2\mu$$

and
$$\lambda_2^+ = \frac{1-2\mu+\sqrt{1+4\mu^2}}{2} > \frac{1-2\mu+(1+2\mu)}{2} = 1.$$
As a result, the fixed point (x_2^*, y_2^*) is unstable. On the other hand, if μ becomes positive (and small), $\sqrt{1+4\mu^2} < \sqrt{1+4\mu+4\mu^2} = 1+2\mu$ and
$$\lambda_2^+ = \frac{1-2\mu+\sqrt{1+4\mu^2}}{2} < \frac{1-2\mu+(1+2\mu)}{2} = 1.$$
Therefore, the stability of the two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) are exchanged, indicating the *transcritical bifurcation* at $\mu = 0$.

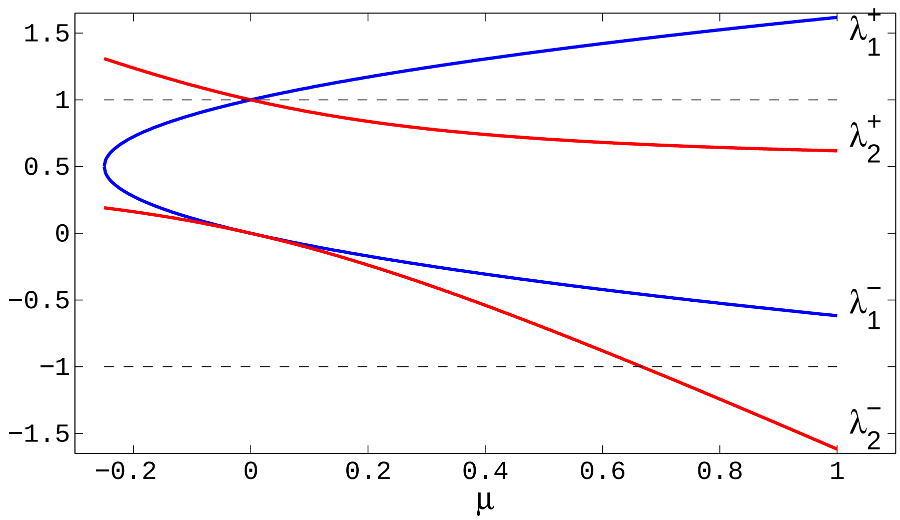


Figure 6.8: The eigenvalues of the Jacobian matrix near the two fixed points $(0, 0)$ and (μ, μ).

The bifurcation is also clear from Figure 6.8. For $\mu \in (-1/4, 0)$, $|\lambda_1^{\pm}| < 1$ and the fixed point $(x_1^*, y_1^*) = (0, 0)$ is stable. The other fixed point $(x_2^*, y_2^*) = (\mu, \mu)$ is stable for $\mu > 0$, but becomes unstable again when $\lambda_2^- = -1$, or $\mu = 2/3$. A period-doubling bifurcation occurs her (associated with eigenvalue -1).

6.5 Other concepts: intermittancy, Lyapunov exponent and the route to chaos

There are other important concepts motivated from maps, like

- Complex iterations from fractals (Julia sets)
- Chaos and its characterisation (sensitive dependence on initial data, existence of "strange attractor", . . .)
- Intermittancy (jumping between nearly periodic and chaotic motions) in chaotic regime
- Lypunov exponents (rate of separation of close trajectories)

These concepts will be briefly mentioned (you will see them more in books for popular audience), but will not appear in the final exam.