5 Bifurcation and centre manifold

For the general ODE $\dot{x} = f(x)$ near its stationary point x^*, we learned early that if none of the eigenvalues of the Jacobian $Df(x^*)$ has a real part zero, then the behaviour of $\dot{x} = f(x)$ is determined by its linearised system $\dot{y} = Df(x^*)y$ with $y \approx x - x^*$. What happens if the Jacobian matrix $Df(x^*)$ has eigenvalues with zero real part?

If some eigenvalues have zero real part, nonlinear terms are expected to play a role, and the behaviour could change accordingly. The study of these qualitative changes in the behaviours (mainly stability/instability of stationary points and periodic orbits), subject to changes in certain parameters, is call *bifurcation theory*. Since the stability/instability of fixed points is indicated precisely by the real part of the eigenvalues, we are going to see how these eigenvalues pass the imaginary axis, as the parameter changes.

Example 5.1. Consider the following two systems

$$(a) \quad \dot{x} = \mu x, \qquad (b) \quad \begin{cases} \dot{x} &= \mu x + \omega y, \\ \dot{y} &= -\omega x + \mu y. \end{cases}$$

It is easy to see that, the eigenvalue $\lambda(\mu) = \mu$ in (a), and $\lambda(\mu) = \mu \pm i\omega$ in (b). The stability is changed when μ crosses zero. More general scenario is shown in Figure 5.1.

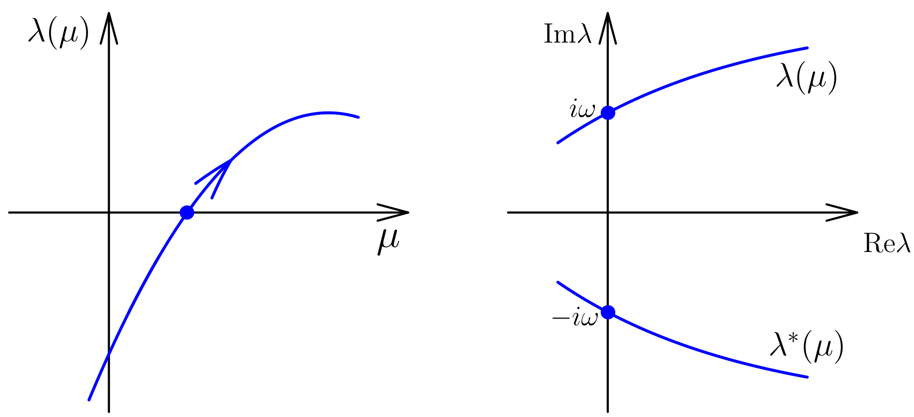


Figure 5.1: Left figure: real eigenvalue passing through zero as a function of μ; Right figure: complex eigenvalues passing through the imaginary axis (think of the eigenvalues as parametrised curves in the complex plane).

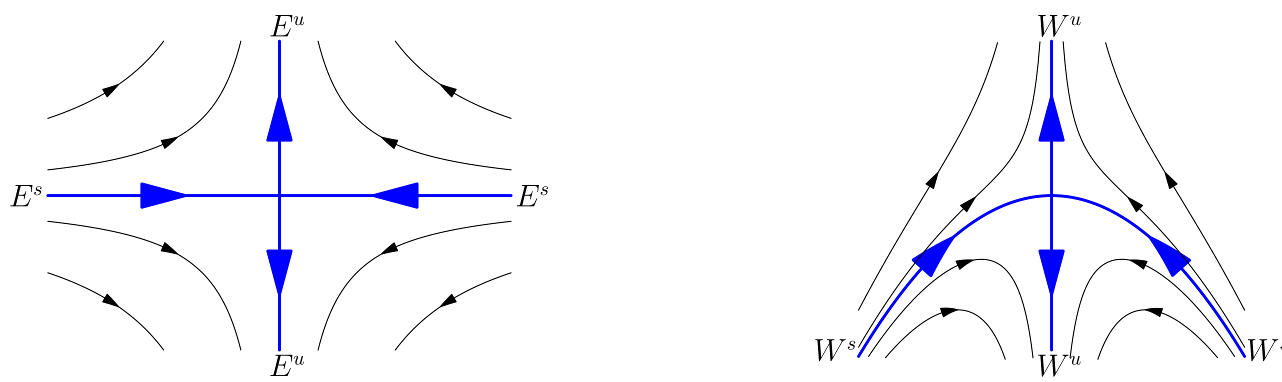
5.1 Centre manifold theorem

We learned *Stable Manifold Theorem* earlier, which states that the structure of the system near a hyperbolic fixed point does not change when nonlinear terms are added. Consider the system

$$\dot{x} = -x, \quad \dot{y} = y + x^2$$

and its linearised system as shown in Figure 5.2.

The stable manifold E^s and the unstable manifold E^u for the linearised system (in normal form) are easy to calculate, which is just the horizontal and vertical axis. Therefore, the



$\dot{x} = -x,\ \dot{y} = y$ $\qquad$ $\dot{x} = -x,\ \dot{y} = y + x^2$

Figure 5.2: The nonlinear system and its linearised system.

notation E^s and E^u (instead of E^s and E^u) is used for the linearised system, to emphasize that they are linear vector spaces. The corresponding stable and unstable manifold for the nonlinear system are usually curved. For the system $\dot{x} = -x, \dot{y} = y + x^2$, the unstable manifold is still the vertical axis (show this!), but the stable manifold is different, and can be approximated as a local series expansion

$$W^s = \{(x, y) \mid y = M(x) = a_2 x^2 + a_3 x^3 + \cdots\}. \tag{5.1}$$

The constant term in $M(x)$ vanishes because W^s passes the origin, and the linear term vanishes because W^s should be tangent to E^s (the horozontal axis), the stable manifold of the linearised system. Now the coefficients $a_2, a_3, \cdots$ representing W^s can be obtained by taking the derivative of both sides of $y = M(x)$. On one hand

$$\dot{y} = y + x^2 = (a_2 + 1)x^2 + a_3 x^3 + \cdots .$$

On the other hand,

$$\frac{d}{dt} M(x) = \left(2a_2 x + 3a_3 x^2 + \cdots\right) \dot{x} = (-x)\left(2a_2 x + 3a_3 x^2 + \cdots\right).$$

Matching the two expressions for $\dot{y} = \frac{d}{dt} M(x)$, we get $a_2 = -1/3, a_3 = a_4 = \cdots = 0$. In other words, the stable manifold is exactly $y = -x^2/3$.

Because the real parts of the eigenvalues are away from zero, the nonlinear system is stable under changes in the parameters or nonlinear terms. However, if there is any eigenvalue with zero real part, we expect some qualitative changes in the property when certain parameter changes, which precisely why bifurcation theory and *Centre Manifold Theorem* are studied together.

Theorem 5.1 (Centre Manifold Theorem)**.** *Given $\dot{x} = f(x), x \in \mathbb{R}^n$, f smooth and suppose $x = 0$ is a stationary point. Suppose the Jacobian matrix $Df(0)$ has eigenvalues in sets σ_u with $\mathrm{Re}(\lambda) > 0$, σ_s with $\mathrm{Re}(\lambda) < 0$ and σ_c with $\mathrm{Re}(\lambda) = 0$ and corresponding generalized linear eigenspaces E^u, E^s and E^c respectively. Then there exist unstable and stable manifolds W^u, W^s of the same dimension as E^u, E^s and tangential to E^s and E^u at $x = 0$; and an invariant centre manifold W^c tangential to E^c at $x = 0$.*

So in general, locally $\mathbb{R}^n = W^c \oplus W^u \oplus W^s$ with the approximate governing equations on *each manifold*

$$\begin{aligned}
\dot{x} &= g(x) && \text{on } W^c \\
\dot{y} &= By && \text{on } W^s && \text{(stable directions)} \\
\dot{z} &= Cz && \text{on } W^u && \text{(unstable directions)},
\end{aligned}$$

where $g(x)$ is quadratic (or higher order) in x, all eigenvalues of B have negative real parts, and all eigenvalues of C have positive real parts.

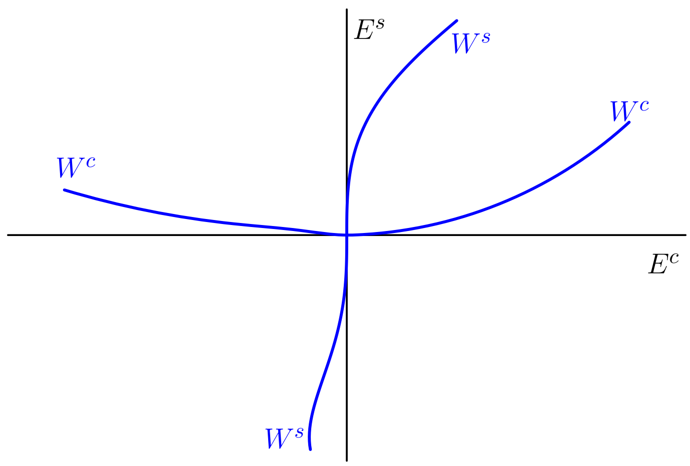


Figure 5.3: Behaviour on W^c depends on nonlinear terms, behaviour off W^c is dominated by exponential contraction in the E^s direction.

In Figure 5.3, there is no unstable direction and in the stable direction the dynamics is attracting, so solutions tend to the centre manifold very quickly. The dynamics on W^c depends on nonlinear terms, is usually much slower and characterise the dynamics of the whole system in the long time. So the question is how this decomposition can be useful in general, and how the centre manifold can be approximated or computed.

5.2 Calculating the centre manifold W^c

Suppose that after a change of coordinate transformation, the hyperplane (or line, if x is one dimension) $(x, 0)$ is spanned by E^c and $(0, y)$ by E^s, then the centre manifold is tangential to $y = 0$ at $(0, 0)$ and we may assume that

$$W^c = \big\{(x, y) \mid y = h(x), h(0) = 0, Dh(0) = 0\big\}.$$

In this coordinate, the system can be written as

$$\dot{x} = Ax + f_1(x, y), \qquad \dot{y} = Cy + f_2(x, y),$$

where all eigenvalues of A have real parts zero and those of C have real parts non-zero, f_i contain only nonlinear terms. So on the centre manifold W^c,

$$\dot{x} = Ax + f_1(x, h(x))$$

and $\dot{y}$ can be calculated on the centre manifold in two ways: directly from the $\dot{y}$ equation above, or by differentiating $y = h(x)$, i.e.

$$\dot{y} = Ch(x) + f_2(x, h(x)) \qquad \text{and} \qquad \dot{y} = \frac{d}{dt}h(x) = Dh(x)\dot{x} = Dh(x)\left[Ax + f_1(x, h(x))\right].$$

where $Dh(x)$ is the (matrix) partial derivatives of $h(x)$, in one dimension simply $h'(x)$.

Expanding h as a Taylor series (noting that the constant and linear terms vanish), the two equations for $\dot{y}$ provide two different polynomials and the coefficients of different monomials can be equated to determine the coefficients of the Taylor expansion.

For a specific problem, here is the general procedure to calculate the centre manifold (which is very similar if you want to find the stable/unstable manifold):

(a) Change the system into normal form (if needed), such that the linearised system is a diagonal matrix

(b) Identify the centre manifold E^c of the linearised system, which is the linear space spanned by the eigenvectors associated with the zero eigenvalues.

(c) Parameterise the centre manifold. You can parameterised E^c first, and then for W^c. For instance, if E^c is the y-axis, then E^c is parameterised as $x = 0$ (and $z = 0$ if in three dimension), and W^c (also a line!) is parameterised by $x = a_2y^2 + a_3y^3 + \cdots$ and $z = b_2y^2 + b_3y^3 + \cdots$. If E^c is the xy-plane in three dimension, then E^c is parameterised by $z = 0$ and W^c is parameterised by

$$z = ax^2 + bxy + cy^2 + \cdots.$$

(d) Finally determine the coefficients in the parmaterisation by differentiation on both sides.

Example 5.2. Consider the system

$$\dot{x} = xy, \quad \dot{y} = -y - x^2.$$

The linear normal form (based on the linearisation at the origin) has the constant matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \text{diag}\,(0, -1).$$

Then the eigenpairs are

$$\lambda_1 = 0, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = -1, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since the matrix A is already in normal form, no coordinate transformation is needed.

Now the centre manifold takes the form

$$y = h(x) = ax^2 + bx^3 + cx^4 + O(x^5), \tag{5.2}$$

There is no constant term, because the centre manifold passes through the origin; there is no linear term, because this manifold should be tangent to e_1 (or equivalently E^c, the centre manifold of the linearised system).

We can determine the coefficients by comparing two ways for calculating $\dot{y}$. Directly from the $\dot{y}$ equation of the system

$$\dot{y} = -y - x^2 = -(ax^2 + bx^3 + cx^4) - x^2 + O(x^5). \tag{5.3}$$

On the other hand, differentiating (5.2) w.r.t t gives $\dot{y} = dh(x)/dt = \dot{x}h'(x)$, i.e.,

$$x(ax^2 + bx^3 + cx^4 + \cdots)(2ax + 3bx^2 + 4cx^3 + \cdots) = 2ax^4 + \cdots . \tag{5.4}$$

Equating coefficients of x^2, x^3 and x^4 in (5.3) and (5.4) gives

$$-a - 1 = 0, \quad -b = 0, \quad -c = 2a^2,$$

i.e. $a = -1$, $b = 0$ and $c = -2$.

Thus the centre manifold is $y = -x^2 - 2x^4 + O(x^5)$ and the dynamics on the centre manifold is

$$\dot{x} = xh(x) = -x^3 - 2x^5 + O(x^7).$$

Thus $\dot{x} < 0$ if $x > 0$ and $\dot{x} > 0$ if $x < 0$. So the origin is stable and the solutions look like a stable node, but the motion onto the centre manifold in the y-direction is much faster than the motion on the centre manifold, leading to a phase portrait as shown in Figure 5.4.

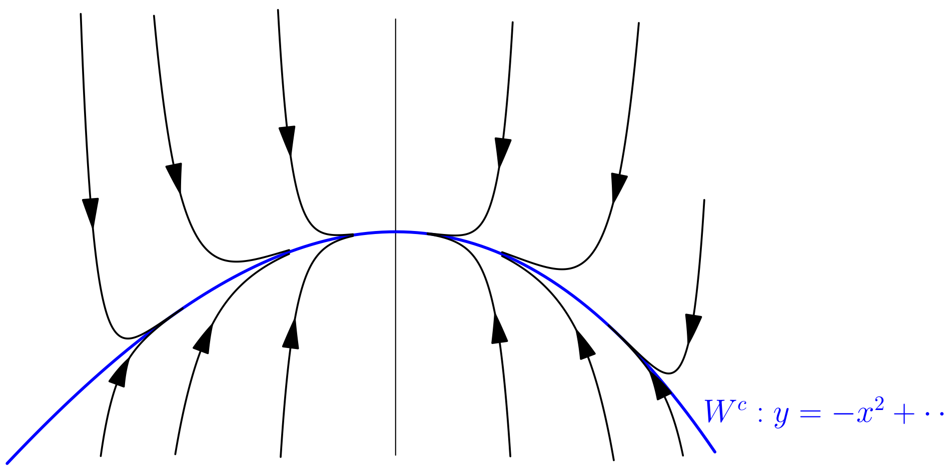


Figure 5.4: Phase portrait showing exponential collapse onto the centre manifold and then slow motion towards $(0, 0)$ on the centre manifold.

Remark. If you try higher order terms, you get

$$y = -x^2 - 2x^4 - 12x^6 - 112x^8 - 1360x^{10} - 19872x^{12} + \cdots .$$

The fast increasing of the coefficients implies that this approximation is valid only in a small neighbourhood of the origin.

Example 5.3. Consider the system

$$\dot{x} = y - x + xy, \qquad \dot{y} = x - y - x^2.$$

First we have to convert the linearised system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

into normal form. It is easy to calculate the eigenpairs of A,

$$\lambda_1 = 0, \quad e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -2, \quad e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let

$$U = [e_1, e_2]^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

the change of variable from (x, y) to (u, v) (in normal form) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+y)/2 \\ (y-x)/2 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = U^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = [e_1, e_2]\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u-v \\ u+v \end{pmatrix}.$$

The new system in (u, v) is

$$\dot{u} = uv - v^2, \dot{v} = -2v + uv - u^2.$$

The centre manifold is parametrised by $v = h(u) = au^2 + bu^3 + cu^4 + \cdots$, then

$$\begin{aligned}
\dot{v} &= (2au + 3bu^2 + 4cu^3 + \cdots)\dot{u} \\
&= (2au + 3bu^2 + 4cu^3 + \cdots)\big(u(au^2 + bu^3 + cu^4 + \cdots) - (au^2 + bu^3 + cu^4 + \cdots)^2\big) \\
&= 2a^2u^4 + \cdots
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\dot{v} &= -2v + uv - u^2 \\
&= -2(au^2 + bu^3 + cu^4 + \cdots) + u(au^2 + bu^3 + cu^4 + \cdots) - u^2 \\
&= (2a+1)u^2 + (2b-a)u^3 + (3c-b)u^4 + \cdots.
\end{aligned}$$

Comparing the coefficients of u^2, u^3 and u^4 of the two expressions of $\dot{v}$, we get

$$a = -\frac{1}{2}, \quad b = -\frac{1}{4}, \quad c = -\frac{3}{8}, \qquad \text{or} \qquad v = -\frac{1}{2}u^2 - \frac{1}{4}u^3 - \frac{3}{8}u^4 + \cdots.$$

The dynamics on the centre manifold is

$$\dot{u} = uv - v^2 = -\frac{1}{2}u^3 - \frac{1}{4}u^4 - \frac{3}{8}u^5 + \cdots,$$

which is stable if u is small. Going back to the original coordinates, the centre manifold is approximately

$$y - x = -\frac{1}{4}(x+y)^2 - \frac{1}{16}(x+y)^3 - \frac{3}{64}(x+y)^4 + \cdots$$

Remark. The calculation could be quite involved if you are calculating unnecessary higher order terms than needed in the end. In general, the lowest power appears in equations with stable linear part. For instance, the second expression above starts with u^2, while the first expression starts with u^4.

Remark. Strictly speaking, the change of variables from (x, y) to (u, v) is not necessary, but we need to know that the centre manifold is represented as

$$y - x = a_2(x+y)^2 + a_3(x+y)^3 + \cdots.$$

Take the derivative of both sides (w.r.t t),

$$\dot{y} - \dot{x} = \big(2a_2(x+y) + 3a_3(x+y)^2 + \cdots\big)(\dot{x} + \dot{y}).$$

After substituting $\dot{x}$ and $\dot{y}$, we compare the coefficients of powers of $(x+y)$ on both sides, and we should get the same answer. (Probably it is worth the effect to perform the change of variable at the very beginning).

5.3 Extended centre manifold

As it stands, the CMT does not allow us to deal with parameters. To include the effect of parameters and hence to treat bifurcations, we work on the extended centre manifolds by augmenting the equation with the apparently trivial equation $\dot{\mu} = 0$:

$$\begin{aligned} \dot{x} &= Ax + f_1(x, y, \mu), \\ \dot{y} &= Cy + f_2(x, y, \mu), \\ \dot{\mu} &= 0. \end{aligned}$$

The additional equation allows us to parametrise the centre manifold as $y = h(x, \mu)$ instead of the form $y = h(x)$ considered in the last section (hence the *extended* centre manifold).

The trivial equation $\dot{\mu} = 0$ adds one more dimension to the centre manifold and allows us to work in a neighbourhood of both $(x, y) = (0, 0)$ in phase space *and* $\mu = 0$ in parameter space, where $\mu = 0$ is the value at which the bifurcation occurs. So A has the zero real part eigenvalues and C has stable and unstable manifolds, and f_1, f_2 contain only nonlinear terms. The CMT gives the motion on the stable and unstable manifolds,W^s and W^u in y, and there is a $n_c + 1$ dimensional centre manifold (where n_c is the dimension of x), valid for $|x|$ and $|\mu|$ small.

This time, if coordinates are chosen so that the central motion is in normal form, the extended centre manifold can be parametrised by $y = h(x, \mu)$, with

$$h(0, 0) = 0, \qquad Dh(0, 0) = 0.$$

Notice that $Dh = [D_x h, D_\mu h]$, which is the partial derivative w.r.t both variables. Then $\dot{x} = Ax + f_1(x, h(x, \mu), \mu)$ is the equation on the (extended) centre manifold.

There are three typical equations (to leading order) on the extended centre manifold if $A = 0$ and x is a scalar:

$$\begin{aligned} &\dot{x} = \mu - x^2 && \text{(saddlenode bifurcation)} \\ &\dot{x} = \mu x - x^2 && \text{(transcritical bifurcation)} \\ &\dot{x} = \mu x - x^3 && \text{(pitchfork bifurcation)} \end{aligned} \tag{5.5}$$

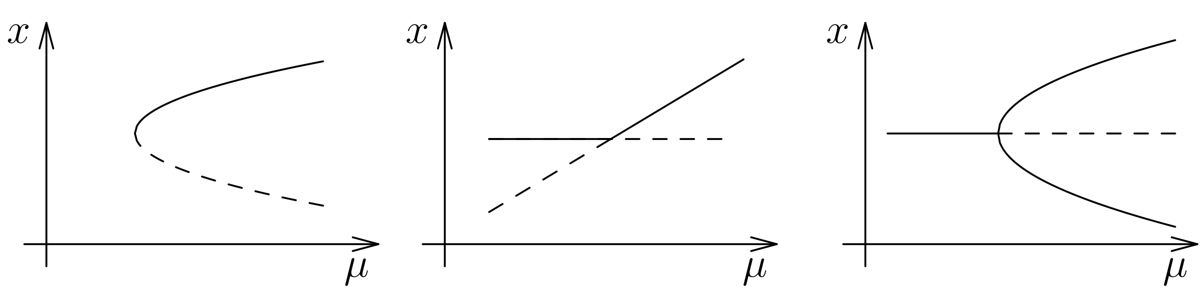


Figure 5.5: (μ, x) plane for local bifurcations of stationary points: saddlenode, transcritical and pitchfork (supercritical).

Typical behaviour is sketched in the (x, μ) plane: these are called *bifurcation diagrams.* By convention, dotted lines are used to show unstable solutions and continuous lines for stable solutions. The pitchfork illustrated her is *supercritical*, meaning that the non-trivial stationary points are stable; if they were unstable then it would be a *subcritical* pitchfork bifurcation (a *subcritical* pitchfork bifurcation when the solid line is dashed, and the dashed line is solid). More details will be given in the next subsection.

Example 5.4. Consider the second order ODE

$$\ddot{u} + \dot{u} - \mu u + u^2 = 0$$

with a parameter μ. Setting $v = \dot{u}$, we get the equivalent system of ODEs

$$\dot{u} = v, \quad \dot{v} = -v + \mu u - u^2. \tag{5.6}$$

At the origin, the matrix for the linear part is $\begin{pmatrix} 0 & 1 \\ \mu & -1 \end{pmatrix}$. The eigenvalues satisfies $s(s+1) - \mu = 0$, so there is a eigenvalue with real part zero, if and only if $s = 0$, or $\mu = 0$. Therefore we expect a bifurcation at the origin if $\mu = 0$.

Rough Explanation of what happens for μ small: The stationary points are governed by $v = 0$, $\mu u - u^2 = 0$. That is $u = 0$, or $u = \mu$, and we expect a transcritical bifurcation (exchange of stability).

The general procedure: (a) Transform to normal form (including in the $\dot{\mu} = 0$ equation); (b) Expand extended CM; (c) Calculate dynamics on CM.

(a). Transformation: The linear part (5.6) at the origin if $\mu = 0$ is not in normal (diagonal) form. We have the eigenpairs,

$$\lambda_1 = 0,\ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -1,\ e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence the change of coordinate uses the matrix of eigenvectors and the NEW coordinates x, y are defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Hence the coordinate transform is

$$x = u + v, \quad y = -v \qquad \text{or} \qquad u = x + y, \quad v = -y.$$

In terms of these new coordinates, the system (5.6) becomes

$$\dot{x} = \dot{u} + \dot{v} = v - v + \mu u - u^2 = \mu(x+y) - (x+y)^2,$$
$$\dot{y} = -\dot{v} = v - \mu u + u^2 = -y - \mu(x+y) + (x+y)^2.$$

(b). Extended Centre Manifold Now the extended system in the new coordinates is

$$\dot{x} = \mu(x+y) - (x+y)^2, \quad \dot{y} = -y - \mu(x+y) + (x+y)^2, \quad \dot{\mu} = 0.$$

The extended centre manifold should be tangential to the (x, μ) plane (or $y = 0$) at $(x, y, \mu) = (0, 0, 0)$, and is parametrised by

$$y = h(x, \mu) = ax^2 + bx\mu + c\mu^2 + h.o.t.$$

From the $\dot{y}$ equation

$$\dot{y} = -(ax^2 + bx\mu + c\mu^2 + \dots) - \mu(x + \dots) + x^2 + \cdots = (1-a)x^2 - (b+1)x\mu + \cdots$$

From the definition of the extended centre manifold

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \mu}\dot{\mu} = (2ax + b\mu)\dot{x} = \cdots ,$$

where all the terms are at least cubic. So equating coefficients of the quadratic terms (of which there are none in the second equation!) gives $a = 1$, $b = -1$, and the extended centre manifold is

$$y = x^2 - x\mu + \dots .$$

(c). Dynamics on the centre manifold. Locally on the extended Centre Manifold $\dot{\mu} = 0$ is trivial so it is the $\dot{x}$ equation that is interesting:

$$\begin{aligned}\dot{x} &= \mu(x + x^2 - \mu x + \dots) - (x^2 + 2x(x^2 - \mu x) + \dots) \\ &= \mu x - x^2 + O(x^3)\end{aligned}$$

Substituting back into the equation for $\dot{x}$ we get (to leading order)

$$\underbrace{\dot{x} = \mu x - x^2}_{\text{Standard Form for transcritical}} \qquad + O(x^3).$$

The phase portrait for the reduced dynamics for x is shown in Figure 5.6 and the phase portrait for the original system is in Figure 5.7.

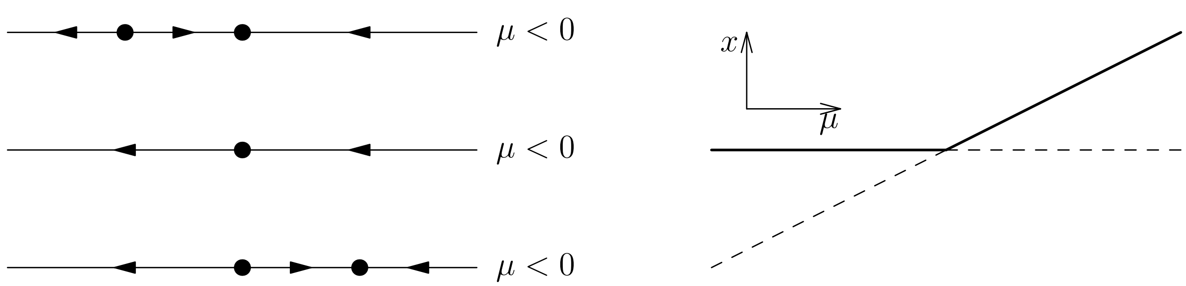


Figure 5.6: Phase portraits on the (one-dimensional) centre manifold and the bifurcation diagram.

Remark. If the stable manifold is of higher dimension, then $y_1 = h_1(x, \mu), y_2 = h_2(x, \mu)$ and we need to find h_1, h_2 using the same method. For example, for the system

$$\dot{x} = \mu x - yz, \quad \dot{y} = -y + x^2, \quad \dot{z} = -z + x^3.$$

Add $\dot{\mu} = 0$ to this system, then the stable manifold expanded by y and z is parameterised by x and μ, that is

$$y = h_1(x, \mu) = a_1x^2 + a_2x\mu + a_3\mu^2 + \cdots , \quad z = h_2(x, \mu) = b_1x^2 + b_2x\mu + b_3\mu^2 + \cdots .$$

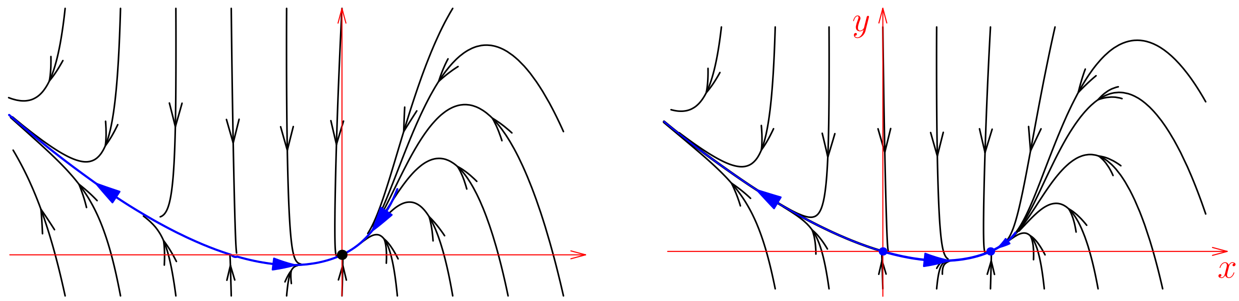


Figure 5.7: Full phase portraits of the dynamics in $\mu < 0$ and $\mu > 0$.

Then $a_1 = 1, a_2 = a_3 = b_1 = b_2 = b_3 = 0$. That is $y = x^2 + \cdots$, but we have to go to cubic polynomials to find the stable manifold for z, which gives $z = x^3 + \cdots$. Therefore, the reduced dynamics on the stable manifold is

$$\dot{x} = f_\mu(x) = \mu x - x^5.$$

If $\mu < 0$, $x = 0$ is the only stable fixed point. If $\mu > 0$, there are three fixed point $0, \mu^{1/4}$ and $-\mu^{1/4}$. Since

$$f'_\mu(0) = \mu > 0, \quad f'_\mu(\pm\mu^{1/4}) = \mu - 5(\pm\mu^{1/4})^4 = -4\mu < 0,$$

the fixed point 0 is unstable, and the fixed points $\pm\mu^{1/4}$ are stable.

5.4 Classifications of bifurcations

Suppose $x = 0 \in \mathbb{R}^n$ is a stationary point of the system of ODEs $\dot{x} = f(x, \mu)$ if $\mu = 0$, and $D_x f(0,0)$ has a single zero eigenvalue. (If the stationary point is x^* at μ^*, then we simply work in shifted coordinates $x - x^*$ and $\mu - \mu^*$). Now we consider the extended centre manifold for the system governed by ($x \in \mathbb{R}$)

$$\dot{x} = f(x, \mu), \qquad \dot{\mu} = 0$$

where f satisfies $f(0,0) = 0$ and $f_x(0,0) = 0$. Consider the Taylor series expansion of f for $|x|, |\mu|$ small:

$$\dot{x} = f(0,0) + f_x(0,0)x + f_\mu(0,0)\mu + \frac{1}{2!}(f_{xx}x^2 + f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu) + O(|x|^3, |\mu|^3)$$

where all partial derivatives are evaluated at $(0,0)$.

By the assumption that $f(0,0) = 0$ ($x = 0$ is the stationary point on the centre manifold for $\mu = 0$) and $f_x(0,0) = 0$ (there is a zero eigenvalue), the above Taylor series is simply

$$\dot{x} = f_\mu(0,0)\mu + \frac{1}{2}\left(f_{xx}(0,0)x^2 + f_{\mu\mu}(0,0)\mu^2 + 2f_{x\mu}(0,0)x\mu\right) + \cdots.$$

Different bifurcations could occur, depending on whether the partial derivatives vanish or not.

Saddle-node Bifurcation $\dot{x} = \mu - x^2$

If both $f_\mu(0,0)$ and $f_{xx}(0,0)$ are non-zero, then

$$\dot{x} = f_\mu(0,0)\mu + \frac{1}{2} f_{xx}(0,0)x^2 + O(|x\mu|, |\mu|^2, \cdots) \approx \mu f_\mu(0,0) + \frac{x^2}{2} f_{xx}(0,0).$$

The stationary points are

$$x^*_\pm \approx \pm\sqrt{-\frac{2f_\mu(0,0)}{f_{xx}(0,0)}\mu} \tag{5.7}$$

if $\mu f_\mu(0,0)/f_{xx}(0,0) \leq 0$. So the stability is determined for sufficiently small $|x|$ and $|\mu|$ by the sign of f_{xx} and f_μ: there is no solution, if $\mu f_\mu / f_{xx} > 0$, and there are two solutions given by (5.7) if $\mu f_\mu / f_{xx} \leq 0$. Since

$$\left.\frac{\partial}{\partial x} f(x,\mu)\right|_{x=x^*_\pm} \approx x^*_\pm f_{xx}(0,0) = \pm\sqrt{-\frac{2f_\mu(0,0)}{f_{xx}(0,0)}\mu} f_{xx}(0,0).$$

Therefore, if $\mu f_\mu / f_{xx} > 0$, x_+ is stable, x_- unstable if $f_{xx} < 0$ and x_- is stable, x_+ unstable if $f_{xx} > 0$ This is a *saddle-node* bifurcation, also called tangential bifurcation or fold bifurcation.

Transcritical Bifurcation $\dot{x} = \mu x - x^2$

If in addition to $f(0,0) = f_x(0,0) = 0$, $f_\mu(0,0)$ is zero, but $f_{xx}(0,0) \neq 0$, the ODE equation becomes

$$\dot{x} \approx \frac{1}{2}\Big(f_{xx}(0,0)x^2 + 2f_{x\mu}(0,0)x\mu + f_{\mu\mu}(0,0)\mu^2\Big)$$

Then the possible stationary points are $x^*_\pm = k_\pm \mu$, where

$$k_\pm = \frac{-f_{x\mu} \pm \sqrt{f^2_{x\mu} - f_{xx}f_{\mu\mu}}}{f_{xx}}.$$

So if $f^2_{x\mu} - f_{xx}f_{\mu\mu} > 0$, there are two branches of solutions which intersect at the bifurcation point $(0,0)$. This is a *transcritical* bifurcation. Stability is determined by looking at the leading order terms of the derivative $f_x(x,\mu)$ and a relatively simple manipulation shows that one branch is stable and the other is unstable, with stability being *exchanged* as μ passes through zero. To show the stability,

$$\left.\frac{\partial}{\partial x} f(x,\mu)\right|_{x=x^*_\pm} = f_{xx}(0,0)x^*_\pm + f_{x\mu}(0,0)\mu = \pm\mu\sqrt{f^2_{x\mu} - f_{xx}f_{\mu\mu}}.$$

So the fixed point x_+ is stable if $\mu < 0$ and unstable if $\mu > 0$; x_- has the opposite stability property.

Pitchfork Bifurcation $\dot{x} = \mu x - x^3$

If $f_\mu(0,0) = f_{xx}(0,0) = 0$ then

$$\dot{x} \approx \frac{1}{2}\left(f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu\right) + \frac{1}{6}\left(f_{xxx}x^3 + f_{\mu\mu\mu}\mu^3 + \dots\right). \tag{5.8}$$

If $f_{x\mu} \neq 0$, there is one branch of solutions with $x \approx -\frac{f_{\mu\mu}}{2f_{x\mu}}\mu$. However there is a second set of solutions by balancing the second term $f_{x\mu}x\mu$ and the third terms $f_{xxx}x^3$:

$$f_{x\mu}x\mu + \frac{1}{6}f_{xxx}x^3 = 0$$

from which, provided $f_{xxx} \neq 0$,

$$x^2 = -\frac{6f_{x\mu}}{f_{xxx}}\mu \tag{5.9}$$

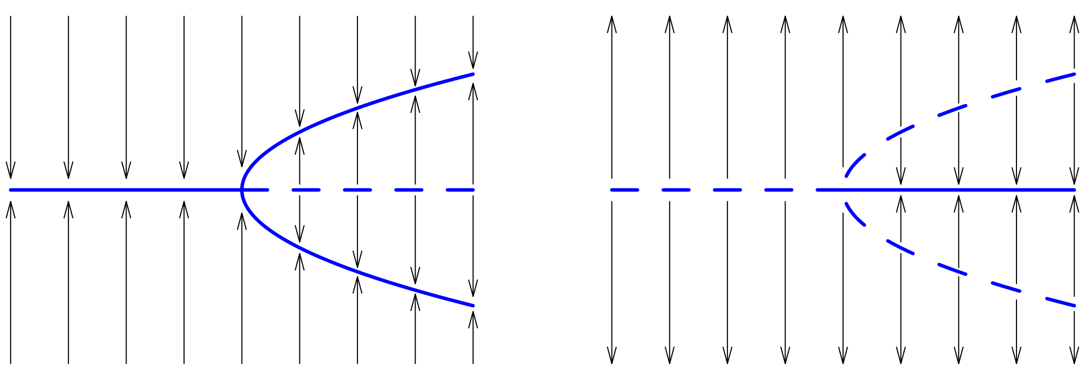
giving two new solutions in whichever sign of μ makes the right hand side positive. There are no other ways of balancing leading order terms (by posing $x \sim \mu^\alpha$) so these are the only bifurcating solutions. Since

$$\frac{\partial}{\partial x}f(x,\mu) = f_{x\mu}\mu + \frac{1}{2}f_{xxx}x^2 + \cdots, \tag{5.10}$$

we see that the solution $x \approx -\frac{f_{\mu\mu}}{2f_{x\mu}}\mu$ is stable (locally) if $f_{x\mu}\mu < 0$ and unstable if $f_{x\mu}\mu > 0$. So the sign of $f_{x\mu}$ determines on which side of $\mu = 0$ this branch is stable.

The stability of second set of solutions is determined by substituting (5.9) into (5.10) giving $-2f_{x\mu}\mu$ and so the stability is the opposite of the simple branch described above.

This is called a *pitchfork* bifurcation: if the non-trivial branch is stable it is called a *supercritical* pitchfork bifurcation and if the non-trivial branch is unstable it is called a *subcritical* pitchfork bifurcation, as shown in Figure 5.8.



Supercritical Pitchfork Bifurcation $\dot{x} = \mu x - x^3$ Subcritical Pitchfork Bifurcation $\dot{x} = -\mu x + x^3$

Figure 5.8: Two types of Pitchfork Bifurcation

Remark. It should be noted that the classification of bifurcation is based on the behaviour near the bifurcation point: in saddle-node bifurcation, the number of fixed points is from zero to two, one stable and one unstable; in transcritical bifurcation, two fixed points always exist and exchange stability; in pitchfork bifurcation, the number of fixed points changes from one to three, and the stability is exchanged.

5.5 Hopf bifurcations

The only 'typical' case not treated in the previous section is the appearance of purely imaginary (simple) eigenvalues $\pm i\omega$, instead of zero eigenvalues. Usually, there is only one zero eigenvalue (as seen in all examples in the previous section), and the centre manifold is only one dimension. But in the simplest case with bifurcation with two purely imaginary eigenvalues, the centre manifold at $\mu = 0$ is two dimensional and the extended centre manifold is three dimensional. The equations governing the bifurcations could very complicated, but a standard example will be enough for our purposes.

Consider the canonical example

$$\dot{x} = \mu x - \omega y - x(x^2 + y^2), \quad \dot{y} = \omega x + \mu y - y(x^2 + y^2)$$

or in polar coordinates $r = \sqrt{x^2 + y^2}, \theta = \arctan y/x$

$$\dot{r} = \mu r - r^3, \quad \dot{\theta} = \omega. \tag{5.11}$$

The linearisation about the origin is

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

with eigenvalues $\mu \pm i\omega$. So if $\mu < 0$, the origin is stable and if $\mu > 0$ it is unstable. Observe that the $\dot{r}$ equation implies that if $\mu > 0$, $r \to \sqrt{\mu}$, i.e. there is a periodic orbit of radius $\sqrt{\mu}$, which is stable (sketch the right hand side of the $\dot{r}$ equation if this is not obvious).

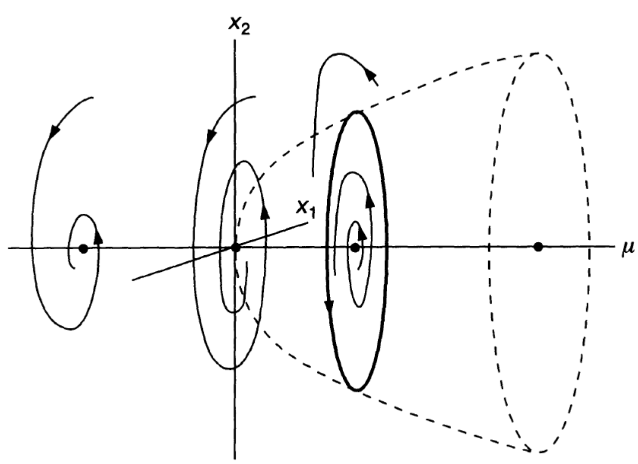


Figure 5.9: Hopf bifurcation: when μ increases, the stable focus becomes unstable, and a periodic solution called *limit cycle* appears.

This is an example of a Hopf bifurcation, also known as Poincaré-Andronov-Hopf bifurcation. As the parameter is varied, a stationary point changes its stability and a periodic orbit is created with the opposite stability (like a pitchfork bifurcation in r). If the bifurcating periodic orbit is stable, then this is a *supercritical Hopf bifurcation* and if it is unstable this is a *subcritical Hopf bifurcation*. The above canonical example can also be written with complex numbers. That is, if we define $z(t) = x(t) + iy(t)$, then the above system is equivalent to

$$\frac{d}{dt} z = (\mu + i\omega)z - z|z|^2.$$

Remark. If only the first equation in (5.11) in the radial variable r is taken, a pitchfork bifurcation happens at $\mu = 0$. This connection (at least qualitatitively) between Hopf bifurcation (for two variables) and pitchfork bifurcation (in radial like variable) is true in general.

Example 5.5 (Brusselator model for autocatalytic reaction)**.** Consider the system of ODEs

$$\dot{x} = a - (b+1)x + x^2 y, \qquad \dot{y} = bx - x^2 y,$$

where a and b are two positive parameters. The unique steady state is $(a, b/a)$ with the Jacobian

$$J = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}.$$

Since the determinant of J is a^2, the only possible bifurcation Hopf bifurcation, which occurs only when the trace is zero. That is when $b^* = 1 + a^2$. For $b < b^*$, the steady state $(a, b/a)$ is stable; for $b > b^*$, it is unstable and a new periodic solution (a limit cycle) appears.

Example 5.6. We considered in Example 4.3 the following model

$$\dot{x} = -x + ay + x^2 y, \qquad \dot{y} = b - ay - x^2 y$$

for glycolysis oscillation with $b = 1/2$ and $a > 0$. For general b, the only stationary point is

$$x^* = b, \qquad y^* = \frac{b}{a + b^2}.$$

We can find the conditions for possible bifurcations. From the Jacobian matrix

$$J(x,y) = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2, \end{pmatrix}$$

we have

$$J(x^*, y^*) = \begin{pmatrix} \frac{b^2-a}{b^2+a} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -a - b^2 \end{pmatrix}$$

Since $\det J(x^*, y^*) = a + b^2 > 0$, the only possible bifurcation is the real parts of the complex eigenvalues pass zero (real eigenvalues can not pass zero, which leads to zero determinant), or Hopf bifurcation. This happens when the trace of $J(x^*, y^*)$ is zero, that is,

$$\mathrm{tr} J(x^*, y^*) = \frac{b^2 - a - a^2 - 2ab^2 - b^4}{a + b^2}.$$

If $\mathrm{tr} J(x^*, y^*) < 0$, the fixed point (x^*, y^*) is stable; otherwise it is not stable.

Example 5.7. Consider the following system

$$\dot{x} = 1 - y^2, \qquad \dot{y} = -x - \mu y - y^2, \tag{5.12}$$

for $\mu \geq 0$.

The fixed points are $(\mu - 1, -1)$ and $(-\mu - 1, 1)$ with the Jacobian matrix is $J = \begin{pmatrix} 0 & -2y \\ -1 & -\mu - 2y \end{pmatrix}$.

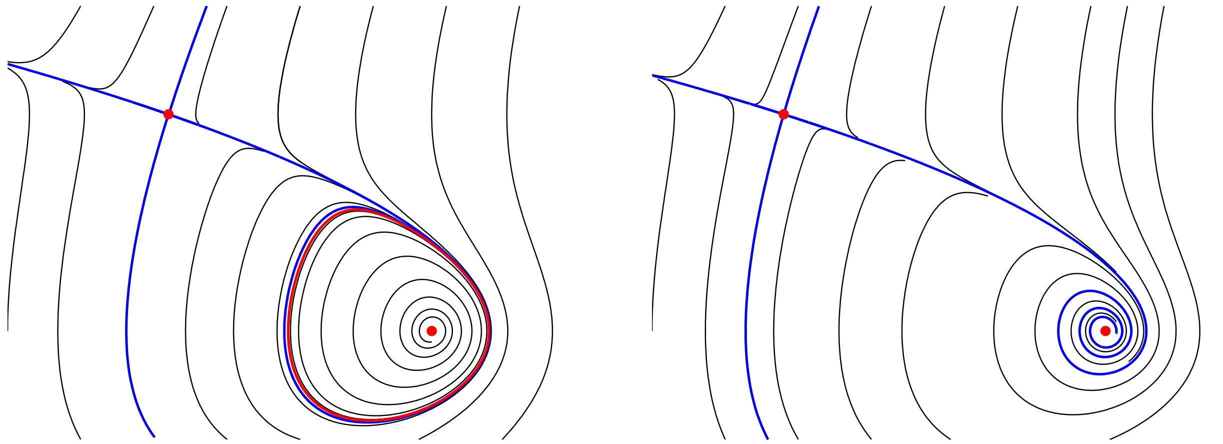


Figure 5.10: The phase portrait of the system (5.12) for $\mu = 1.8$ (left figure) and $\mu = 2.2$ (right right). As μ passes 2, the periodic solution (or the limit cycle) disappear, and the unstable focus at $(\mu - 1, -1)$ becomes a stable focus. (Can you add arrows to indicate the direction of the trajectories, based on the local behaviours of the stationary points?)

At the fixed point $(-\mu - 1, 1)$, $J = \begin{pmatrix} 0 & -2 \\ -1 & -\mu - 2 \end{pmatrix}$. Since $\det J = -2 < 0$, the two eigenvalues have opposite signs, and this is always a saddle point.

At the fixed point $(\mu - 1, -1)$, $J = \begin{pmatrix} 0 & 2 \\ -1 & -\mu + 2 \end{pmatrix}$ and the eigenvalues are the roots of

$$\lambda^2 + (\mu - 2)\lambda + 2 = 0,$$

or $\lambda_\pm = \frac{2-\mu\pm\sqrt{\mu^2-4\mu-4}}{2}$. Since $\lambda_+\lambda_- = 2 > 0$, the real parts of the eigenvalues pass zero if and only if μ passes 2. At $\mu = 2$, the two eigenvalues are $\pm\sqrt{2}i$, this is a unstable focus becomes a stable focus. When μ continues to increase beyond $1 + 2\sqrt{2}$, the discriminant $\Delta = \mu^2 - 4\mu - 4$ becomes positive, and the two eigenvalues are both real and negative. Therefore, the stable focus becomes a stable node, as shown in Figure 5.10.

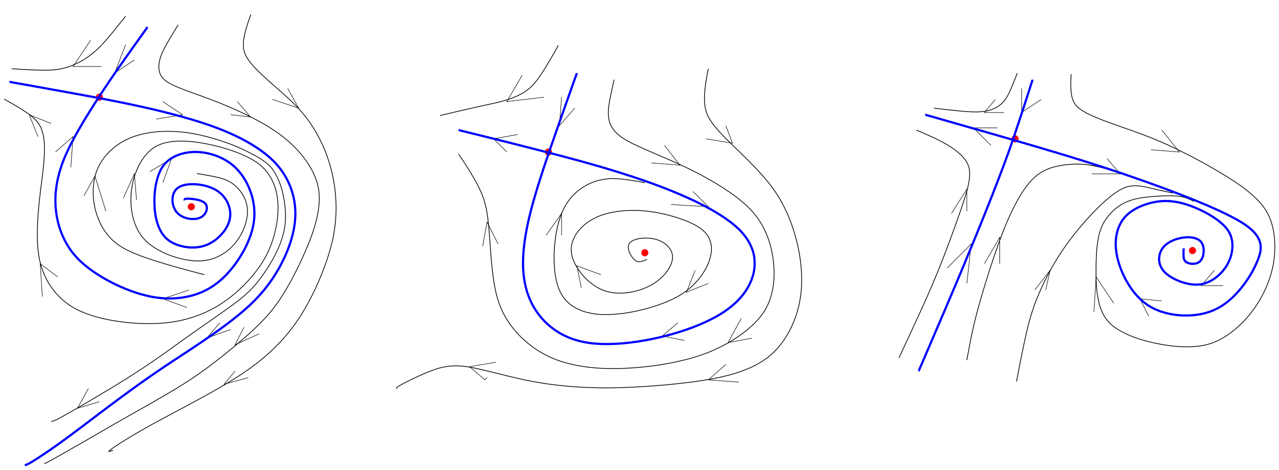


Figure 5.11: The global bifurcation at about $\mu^* = 1.63$, where there is a homoclinic orbit at $(-\mu - 1, 1)$.

Remark. There is actually another bifurcation at about $\mu^* = 1.63$, with a homoclinic orbit at the fixed point $(-\mu - 1, 1)$: the unstable manifold W^u coincides with the stable manifold W^s through this fixed point. This kind of *global bifurcation* is much more difficult to study, where the critical parameter μ^* can not be determined as the local bifurcation in the previous few examples.