4 Periodic orbits

The behaviour of a dynamical system could be very complicated, even we just consider those limiting ones, as chaotic trajectories could appear. But in the plane, the limiting trajectories on a compact set have relative simple structures, consisting only fixed points and closed orbits (that connecting fixed points or periodic). We have talked about fixed points, while periodic orbits are much harder to work with than stationary points, as the explicit expressions are not always available. In this section, we will describe three aspects:

in the plane: Poincaré-Bendixson Theorem (existence of periodic solutions)

linearisation: Floquet Theory (solutions of linear ODEs with periodic coefficients)

and we will show how single periodic orbit could arise from bifurcation.

4.1 Poincaré-Bendixson Theorem

In the plane, there is a classic result called *Poincaré Bendixson Theorem* for proving the existence of periodic orbits. This theorem is stated without proof (can be found in Chapter 6.6 in the book by Meiss), but you need to be able to state the conditions correctly and apply it to examples.

Theorem 4.1 (Poincaré-Bendixson Theorem for the existence of periodic orbits)**.** *Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and f smooth. If there exists a compact (closed and bounded) subset $\mathcal{D} \subset \mathbb{R}^2$ containing no stationary points and $p \in \mathcal{D}$ such that $\varphi_t(p) \in D$ for all $t \geq 0$ (i.e, $\mathcal{D}$ is invariant), then there is at least one periodic orbit in $\mathcal{D}$ and the orbit of p tends to this periodic orbit.*

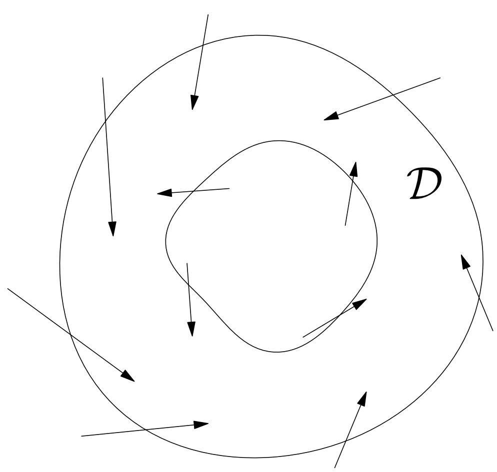

Figure 4.1: Typical Poincaré-Bendixson region.

How do we apply the theorem? In applications the region $\mathcal{D}$ is usually annular (as show in Figure 4.1). The strategy will be to show that solutions enter and do not leave an annular region containing no stationary points, hence we can apply the Poincaré-Bendixson Theorem stated above. This is usually done by constructing a region lying between two closed curves $\partial\mathcal{D}_\pm$ defined by $V_\pm(x) = c_\pm$ with $\partial\mathcal{D}_-$ lying inside $\partial\mathcal{D}_+$ and $V_\pm$ increasing outwards (like Lyapunov functions). If in addition (for example) $\dot{V}_-(x) > 0$ on $V_-(x) = c_-$ and $\dot{V}_+(x) < 0$ on $V_+(x) = c_+$ then trajectories on both boundaries point into the region and trajectories inside the region cannot cross outwards.

Remark. The function V is not unique, and many functions could lead to the same conclusion (of course the annular region $\mathcal{D}$ could be different). However, a good choice of the function V can enormously simplify your calculation, motivated from the corresponding normal form.

Example 4.1. Consider the system

$$\begin{aligned}\dot{x} &= x + y - 4x(x^2+y^2)\\ \dot{y} &= -x + y - 4y(x^2+y^2)\end{aligned}$$

The stationary Point $(0,0)$ is obvious. Are there any others?

From the first equation

$$4x(x^2+y^2) = x+y \quad \text{ i.e. } 4xy(x^2+y^2) = xy + y^2$$

and from the the second equation

$$4y(x^2+y^2) = -x+y \quad \text{ i.e. } 4xy(x^2+y^2) = -x^2 + xy$$

Therefore,

$$4xy(x^2+y^2) = xy + y^2 = -x^2 + xy,$$

which implies that $x^2 + y^2 = 0$, or there is no solutions other than $(0,0)$.

Now consider $V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ (can your think of any motivation for this choice?). Then

$$\begin{aligned}\dot{V} &= x\dot{x} + y\dot{y}\\ &= x(x+y-4x(x^2+y^2)) + y(-x+y-4y(x^2+y^2)\\ &= x^2+y^2-4(x^2+y^2)^2.\end{aligned} \tag{4.1}$$

Define

$$\mathcal{D} = \left\{(x,y) \mid \frac{1}{8} \le x^2+y^2 \le 1\right\}.$$

Then $\dot{V} > 0$ on $x^2+y^2 = \frac{1}{8}$ (which is the set $V(x,y) = \frac{1}{16}$) i.e. solutions enter $\mathcal{D}$ across this boundary; $\dot{V} < 0$ on $x^2+y^2 = 1$ (which is the set $V(x,y) = \frac{1}{2}$) i.e. solutions enter $\mathcal{D}$ across this boundary.

So solutions enter and do not leave $\mathcal{D}$ which is closed, bounded and contains no stationary points, and hence there is at least one periodic orbit in $\mathcal{D}$. (Note that *any* outer boundary bigger than $\frac{1}{4}$ and lower boundary less than this will do.)

Remark. In polar coordinates this example is easy:

$$\dot{r} = r(1-4r^2), \qquad \dot{\theta} = -1.$$

Thus the $\dot{\theta}$ equation shows that $(0,0)$ is the only stationary point, whilst the $\dot{r}$ equation shows $r \to \frac{1}{2}$!

Equivalently we could have noted that the equation for $\dot{V}$ in (4.1) can be written as

$$\dot{V} = 2V(1 - 8V).$$

So solutions $V = \frac{1}{2}r^2 \to \frac{1}{8}$.

If the boundaries are given by $V(x) = c$, $x \in \mathbb{R}^2$, and V increases outwards then $\dot{V} = \nabla V \cdot f(x)$ (by the Chain Rule) and so if $\dot{V} > 0$ on the inner boundary then f points **into** $\mathcal{D}$ on the lower boundary and if $\dot{V} < 0$ on the outer boundary then f also points **into** $\mathcal{D}$ on the outer boundary. The region $\mathcal{D}$ is a trapping region (no solutions escape from $\mathcal{D}$). This can be proved rigorously using the Mean Value Theorem as in sections on Lyapunov functions, but this geometric description is self-evident and does not really need further explication.

What if $\dot{V} \geq 0$ on the outer boundary and $\dot{V} \leq 0$ on the inner boundary (i.e, with non-strict inequalities)? Here the geometric argument does not hold (trajectories can be tangent to the boundary at some places) and we need to work a little harder. If we choose

$$\mathcal{D} = \left\{(x, y) \mid \frac{1}{4} \leq x^2 + y^2 \leq 1\right\}$$

in the previous example, then $\dot{V} \geq 0$ (actually $\dot{V} \equiv 0$) on $\partial\mathcal{D}_- = \{(x, y) \mid x^2 + y^2 = 1/4\}$. In this case $\partial\mathcal{D}_-$ is exactly the periodic orbit.

Example 4.2. Prove existence of a periodic orbit for

$$\dot{x} = y, \qquad \dot{y} = -x + y(1 - 3x^2 - 6y^2). \tag{4.2}$$

First consider stationary points: from $\dot{x} = 0$ we find $y = 0$ and then $\dot{y} = 0$ implies $x = 0$ so $(0, 0)$ is a unique stationary point.

Try $V(x, y) = x^2 + y^2$, then

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2xy + 2y\left[-x + y(1 - 3x^2 - 6y^2)\right] = 2y^2(1 - 3x^2 - 6y^2).$$

So if $1-3x^2-6y^2 < 0$ (i.e. in the neighbourhood of the origin), $\dot{V} \leq 0$, and if $1-3x^2-6y^2 > 0$ (i.e. far away from the origin), $\dot{V} \geq 0$. To apply the Poincaré-Bendixson Theorem, we have to understand how the geometry of these ellipses $\{(x, y) \mid 1 - 3x^2 - 6y^2 = 0\}$ describing the sign of $\dot{V}$ interacts with the geometry of the circles of constant $V(x, y) = x^2 + y^2$ that will be used to define $\mathcal{D}$. Moreover, every such circle contains some points on the x-axis (with $y = 0$), and $\dot{V} = 0$ on some parts of $V = c$. As a result, the simple geometric arguments used above will not hold (we need $\dot{V}$ strictly less or greater than zero.

How does the $\dot{V}$ equation relate to the circles $V = c$? First,

$$\dot{V} = 2y^2(1 - 3x^2 - 6y^2) \leq 2y^2(1 - 3x^2 - 3y^2)$$

So if $x^2 + y^2 = V \geq \frac{1}{3}$, then $\dot{V} \leq 0$.

Let the outer boundary of $\mathcal{D}$ be $x^2 + y^2 = \frac{1}{3}$, call this $\partial\mathcal{D}_+$. Then if $p = (x_0, y_0)$ with $|p| = \sqrt{x_0^2 + y_0^2} \leq \frac{1}{3}$ and $V(p) \leq \frac{1}{3}$, we claim $V(\varphi_t(p)) = |\varphi_t(p)|^2 \leq \frac{1}{3}$ for all $t \geq 0$. Here $\varphi_t(p)$ is the solution to (4.2) starting from p at time $t = 0$ (this form is used to emphasise the dependence of the initial condition on p). Suppose this claim is not true, then there exist two times t_0 and t_1, such that

$$V(\varphi_{t_0}(p)) = \frac{1}{3}$$

and

$$V(\varphi_t(p)) > \frac{1}{3} \quad \text{for all} \quad t \in (t_0, t_1].$$

In particular

$$V(\varphi_{t_1}(p)) - V(\varphi_{t_0}(p)) > 0.$$

Hence by the Mean Value Theorem,

$$V(\varphi_{t_1}(p, t_1)) - V(\varphi_{t_0}(p)) = \dot{V}(\varphi_t(p))(t_1 - t_0) \quad \text{for some} \quad t \in [t_0, t_1].$$

On the other hand,

$$\frac{dV(\varphi_t(p)}{dt} \le 0 \text{ for all } t \in [t_0, t_1]\,.$$

Therefore, $0 < V(\varphi_{t_1}(p, t_1)) - V(\varphi_{t_0}(p)) = \dot{V}(\varphi_t(p))(t_1 - t_0) \le 0$, which is a contradiction.

Now we consider the inner boundary. The fact

$$\dot{V} = 2y^2(1 - 3x^2 - 6y^2) \ge 2y^2(1 - 6x^2 - 6y^2)$$

implies that $\dot{V} \ge 0$ on the set $\{(x, y) \mid x^2 + y^2 \le 1/6\}$, and motivates the choice of the inner boundary

$$\partial\mathcal{D}_- = \left\{(x, y) \mid x^2 + y^2 = \frac{1}{6}\right\}.$$

We claim that if $V(p) \ge \frac{1}{6}$ then $V(\varphi_t(p)) \ge \frac{1}{6}$ for all $t \ge 0$. The proof is almost exactly the same as the above case (try it!).

So if we choose $\mathcal{D} = \left\{(x, y) \mid \frac{1}{6} \le V(x, y) \le \frac{1}{3}\right\}$, a solution that starts in $\mathcal{D}$ stays in $\mathcal{D}$. Moreover, $\mathcal{D}$ contains no stationary points and hence by the Poincaré-Bendixson Theorem there is at least one periodic orbit in $\mathcal{D}$.

Now we consider a much more complicated example, in which the Poincaré-Bendixson region is much more difficult to construct.

Example 4.3 (Glycolysis oscillation)**.** In this system of ODEs modelling the process how the human body converts glucose (sugar) into energy, x is the ADP concentration and y is the F6P (fructose-t-phosphate) concentration ($a > 0$):

$$\dot{x} = -x + ay + x^2y, \qquad \dot{y} = \frac{1}{2} - ay - x^2y.$$

We want to show that there are oscillations (periodic orbits) if the positive parameter a is sufficiently small. Start by considering stationary points and their stability.

Stationary Points: Solving the system

$$x^2y = x - ay, \qquad x^2y = \frac{1}{2} - ay,$$

we get $x^* = \frac{1}{2}$ and $y^* = \frac{1}{2(a+x^2)} = \frac{2}{4a+1}$. So $\left(\frac{1}{2}, \frac{2}{4a+1}\right)$ is the only stationary point.

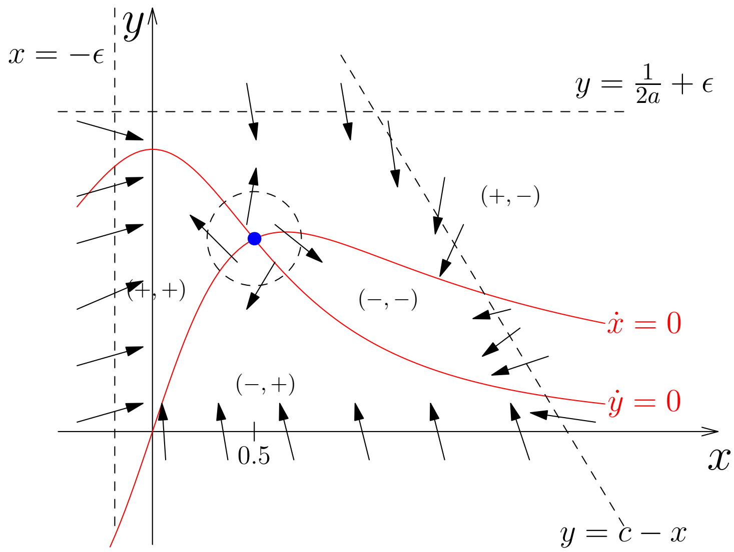

Figure 4.2: Poincaré-Bendixson region for the glycolysis model.

Stability: The Jacobian matrix is

$$J = \begin{pmatrix} -1 + 2xy & x^2 + a \\ -2xy & -(x^2 + a) \end{pmatrix}.$$

At the stationary point $\left(\frac{1}{2}, \frac{2}{4a+1}\right)$,

$$J(x^*, y^*) = \begin{pmatrix} -1 + y^* & \frac{1}{2y^*} \\ -y^* & -\frac{1}{2y^*} \end{pmatrix}.$$

Therefore, we get the determinant and the trace

$$\det J = \frac{1}{2y^*} > 0, \quad \mathrm{tr} J = -1 + y^* - \frac{1}{2y^*}.$$

Since the product of the roots ($\det J$) is positive the stationary point is not a saddle, and so it is stable if Tr $J < 0$ and unstable if Tr $J > 0$. In other words, the stationary point is unstable if $1 + \frac{1}{2y^*} - y^* < 0$ or

$$1 + \frac{4a+1}{4} - \frac{2}{4a+1} < 0.$$

This equation can be written as

$$4(4a+1) + (4a+1)^2 - 8 < 0 \qquad \text{or} \qquad 16a + 4 + 16a^2 + 8a + 1 - 8 < 0$$

which is simplified as

$$3 - 24a - 16a^2 > 0.$$

Thus if a is sufficiently small, then the fixed point is unstable and there is a small closed curve containing the fixed point that solutions cross outwards.

Constructing a PB region(when $a > 0$ is sufficiently small): As noted earlier, if a is sufficiently small then the stationary point $(\frac{1}{2}, \frac{2}{1+4a})$ is unstable and so we can use the Lyapunov function constructed from adjoint eigenvectors in reverse time to construct an inner boundary that trajectories cross outwards. In what follows $\epsilon > 0$ is a small constant.

The remainder of the Poincaré-Bendixson region will be constructed using four straight lines (see Figure 4.2): the x-axis, a vertical line near the y-axis, a horizontal line at larger y and a part of $x + y = c$, with c chosen later. The first two are obvious. If $y = 0$ (the x-axis) then $\dot{y} = 1/2 > 0$ and so trajectories cross the x-axis upwards into $y > 0$. If $x = -\epsilon < 0$ (near the y-axis) then $\dot{x} = \epsilon + ay$ and so $\dot{x} > 0$ if $y > 0$. Thus solutions cross the x-axis and a horizontal line near the y-axis into the positive quadrant formed by the two lines. (The use of ϵ is so that we don't have to worry about the behaviour at the origin, where $\dot{x} = 0$ had we chosen the two coordinate axes.)

To think about the remaining straight lines of the boundary, draw the two nullclines:

$$y = \frac{x}{a + x^2} \qquad \text{on which } \dot{x} = 0,$$
$$y = \frac{1}{2(a + x^2)} \qquad \text{on which } \dot{y} = 0.$$

In other words, the direction field (or the vector field) is vertical on the line $y = x/(a + x^2)$ and horizontal on $y = 1/2(a + x^2)$.

Thus the nullcline $y = 1/2(a + x^2)$ for $\dot{y}$ has a maximum at $y = \frac{1}{2a}$; if $y > \frac{1}{2a}$, then $\dot{y} < 0$ and trajectories have a component downwards. This suggests using the line

$$y = \frac{1}{2a} + \epsilon$$

which all solutions cross downwards.

But we can not choose a vertical line $x = M$ for some large M to complete the closed region, because of the term x^2y ($\dot{x}$ may not be negative on this vertical line). However, we note that the sign for $\dot{x} + \dot{y} = -x + 1/2$ is much simpler (negative when $x > 1/2$). This motivate the choice of straight lines of the form $x + y = c$ for some c. If we choose $F(x, y) = x + y - c$ and define the region $D = \{(x, y) \mid F(x, y) < 0$, then $\dot{F} < 0$ if $x > 1/2$. In other words, trajectories on the line $x + y = c$ cross inwards to D.

The choice of c is determined by closing the outer boundary as simply as possible: choose it so that $x + y = c$ intersects $y = \frac{1}{2a} + \epsilon$ when $x = \frac{1}{2}$, i.e.

$$c = \frac{1}{2} + \frac{1}{2a} + \epsilon.$$

So consider region (see Figure 4.2) bounded by

(a) $x = 0, \qquad 0 \le y \le \dfrac{1}{2a} + \epsilon$

(b) $y = 0, \qquad 0 \le x \le \dfrac{1}{2} + \dfrac{1}{2a} + \epsilon$

(c) $y = \dfrac{1}{2a} + \epsilon, \qquad 0 \le x \le \dfrac{1}{2}$

(d) $x + y - (\frac{1}{2} + \frac{1}{2a} + \epsilon) = 0, \qquad \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{2a} + \epsilon$

Solutions enter but do not leave this region bounded by these curves. Together with the small closed curve about the unstable (a sufficiently small) stationary point which solutions cross outwards, this forms a Poincaré-Bendixson region. Hence there exists a periodic orbit.

Theorem 4.2 (Non-existence of periodic solution)**.** *For the system of ODEs*

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

there is no periodic solution in any simply-connected domain of the phase plane, if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ *does not change sign.*

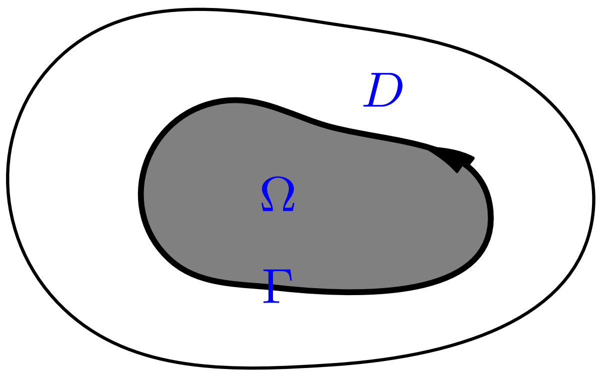

Figure 4.3: A periodic orbit Γ inside the simply-connected domain D, and the region Ω bounded by Γ.

Proof. Without loss of generality, we can assume that $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is positive on the domain Ω, inside which there is a periodic orbit Γ, as show in Figure 4.3. Using Green's Theorem,

$$0 < \iint_{\Omega} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dydx = \int_{\Gamma} (fdy - gdx). \tag{4.3}$$

On the other hand, because Γ is an orbit of the system $\dot{x} = f, \dot{y} = g$, we have for (x, y) on Γ,

$$f\frac{dy}{dt} = fg = g\frac{dx}{dt}$$

or $fdy - gdx \equiv 0$ on Γ. Therefore, the equation (4.3) can not be satisfied, and there is no periodic solution. ☐

Remark (Physical intuition). The quantity $\partial_x f + \partial_y g$ is called the *divergence* of the vector field (f, g) associated the system of ODEs (the same as you learned in your elementary calculus class and other courses like Fluid Mechanics). The divergence characterises the local expansion or contraction of the volume element moving with the vector field.

Example 4.4 (Normal form of ODEs with complex eigenvalue)**.** Consider the following canonical form of ODE

$$\begin{cases} \dot{x} &= \rho x + \omega y \\ \dot{y} &= -\omega x + \rho y \end{cases}.$$

There is a periodic solution if and only if $\rho = 0$ (i.e centre) where the divergence of the vector field is 2ρ. For $\rho = 0$, any circle centred at the origin is a periodic solution.

Remark. The above theorem only says no periodic orbit inside the domain Ω bounded by Γ. But periodic orbits could exist that are completely outside Ω, or intersect with Ω or Γ, as shown in the following example.

Example 4.5. Consider the same system of ODEs

$$\dot{x} = x + y - 4x(x^2 + y^2), \qquad \dot{y} = -x + y - 4y(x^2 + y^2)$$

as in Example 4.1. Since the divergence of the vector field is

$$\frac{\partial}{\partial x}\big(x + y - 4x(x^2 + y^2)\big) + \frac{\partial}{\partial y}\big(-x + y - 4y(x^2 + y^2)\big) = 2 - 16(x^2 + y^2).$$

Since the divergence is positive on the set $\Omega = \{(x, y) \mid x^2 + y^2 < 1/8\}$, there is no periodic inside Ω (but there is one periodic orbit outside of Ω).

Example 4.6. The ODE $\ddot{x} + g(x) = 0$ usually describes oscillations (the harmonic oscillator with $g(x) = kx/m$), with the conserved energy $E = \dot{x}^2/2 + \int^x g(x)dx$. A more realistic model with friction is $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Show that if f is of one sign (say $f > 0$, the friction always opposite to the direction of motion), then there is no periodic solution. The latter system is equivalent to

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x),$$

and the divergence is

$$\frac{\partial}{\partial x}y + \frac{\partial}{\partial y}\big(-f(x)y - g(x)\big) = -f(x).$$

Therefore, there is no periodic solutions.

However, if the direction of the friction is of mixed sign, then periodic solutions are possible, like the van der Pol's ODE

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0.$$

4.2 Floquet theory

Suppose $u(t)$ is a periodic solution of $\dot{x} = f(x)$, so $u(t + T) = u(t)$ for all $t \in \mathbb{R}$. What is the linearisation of the equation about this solution? Note that since it *is* a solution u satisfies $\dot{u} = f(u)$. Set $x(t) = u(t) + v(t)$ where $|v| \ll 1$. Then

$$\dot{x} = \dot{u} + \dot{v} = f(u + v) = f(u) + \mathrm{D}f(u)v + O(|v|^2).$$

i.e. $\dot{v} = \mathrm{D}f(u)v$ is the linearisation of the equation about u, where $\mathrm{D}f(u)$ is the $n \times n$ Jacobian matrix evaluated at u. This system of ODEs is linear (in v), but its coefficient is periodic, different from the constant coefficient ODEs we considered earlier. The Floquet theory deals with the structure of such system of ODEs

$$\dot{v} = A(t)v, \quad A(t + T) = A(t).$$

Example 4.7 (One dimensional case)**.** The one-dimensional case is elementary and somewhat misleading if we are thinking about its generalization to higher dimensions. Nonetheless it is worth considering as an exercise. We have

$$\dot{v} = a(t)v, \qquad a(t+T) = a(t),$$

where both $a(t)$ and $v(t)$ are scalar. This ODE is separable, and the solution is given by

$$v(t) = v_0 \exp\left(\int_0^t a(s)\, ds\right).$$

To find the instruction of the solution, we look at $t = nT$ first, that is $v(nT) = v_0 \exp(\int_0^{nT} a(s)\, ds)$. Then because a is periodic,

$$\begin{aligned}\int_0^{nT} a(s)\, ds &= \int_0^T a(s)\, ds + \cdots + \int_{(n-1)T}^{nT} a(s)\, ds \\ &= n\int_0^T a(s)\, ds \qquad \text{as} \qquad a(s+T) = a(s)\end{aligned}$$

This implies that $v(nT) = v_0\Phi^n$ where the constant $\Phi = \exp\left(\int_0^T a(s)\, ds\right)$. Therefore, $|v(nT)|$ is increasing/decreasing (i.e. unstable or stable) according to whether $|\Phi|$ is greater than or less than one. If we write

$$v(t) = w(t)\Phi^{t/T} = w(t)\exp\left(\frac{t}{T}\int_0^T a(s)ds\right), \tag{4.4}$$

then $w(t)$ is a periodic. In fact, we have

$$v(t{+}T) = v(t)\exp\left(\int_t^{t+T} a(s)ds\right) = v(t)\exp\left(\int_0^T a(s)ds\right) = w(t)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right).$$

Comparing with the expression (4.4) (at $t+T$ instead of t), then

$$v(t+T) = w(t+T)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right) = w(t)\exp\left(\frac{t+T}{T}\int_0^T a(s)ds\right),$$

that $w(t) = w(t+T)$ and w is periodic with period T. Therefore, in general the solution v is no periodic, unless $\Phi = 1$.

The nice solution in terms of exponentials does not work in $\mathbb{R}^n$: even though the solution to $\dot{x} = A(t)x$ with initial condition $v(0) = v_0$ can be written as $v(t) = \Phi(t)v_0$, the matrix $\Phi(t)$ depends on the coefficient $A(t)$ in a much more complicated way than that in one dimension. Nevertheless, such a matrix $\Phi(t)$ exists, and plays a similar role of as in one dimension.

Example 4.8 (A one-way coupled linear PDE with periodic coefficient)**.** Consider the following two systems

$$(1) \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & \sin t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; \qquad (2) \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \sin t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where the coefficient $A(t)$ is periodic with period 2π. In both cases, x is governed by the same equation $\dot{x} = -x$ and the general solution is given by $x(t) = x_0 e^{-t}$. For system (1), y is governed by $\dot{y} = x + y\sin t$, whose solution is given by

$$y(t) = x_0 e^{-\cos t}\int_0^t e^{-\tau+\cos\tau}d\tau + y_0 e^{1-\cos t}.$$

For system (2), y is governed by $\dot{y} = x\sin t + y$,

$$y(t) = -\frac{e^{-t}}{5}(\cos t + 2\sin t)x_0 + \left(y_0 + \frac{x_0}{5}\right)e^t.$$

Question: are there any periodic solution with period 2π for either system with particular initial condition?

The Structure of linear ODEs with periodic coefficients. Let v_k be the solution to the system

$$\dot{v} = A(t)v, \qquad A(t+T) = A(t)$$

with the initial condition $v_k(0) = e_k$, the k-th canonical basis in $\mathbb{R}^n$ (the vector with 1 at k-th entry, and 0 otherwise). Because of the linearity, any solution with initial condition $v(0) = (\alpha_1, \cdots, \alpha_n)^t = \alpha_1 e_1 + \cdots + \alpha_n e_n$ is

$$v(t) = \sum_{k=1}^n \alpha_n v_k(t) = \Phi(t)v(0),$$

where $\Phi(t)$ is the so called *fundamental matrix* with v_k at k-th column, i.e,

$$\Phi(t) = \begin{bmatrix} v_1(t) & v_2(t) & \cdots & v_n(t) \end{bmatrix}.$$

In other words, $\Phi(t)$ is the solution to the matrix ODEs $\dot{\Phi} = A(t)\Phi$ starting with the identity matrix as initial condition. We can get from the solutions in Example 4.7 that the fundamental matrices are

$$\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-\cos t}\int_0^t e^{-\tau+\cos\tau}d\tau & e^{1-\cos t} \end{pmatrix}$$

and

$$\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ \frac{e^{-t}}{5}\left(e^t - e^{-t}\cos t\right) - \frac{2}{5}e^{-t}\sin t & e^t \end{pmatrix}$$

respectively. There is no need to start with $v_k(0) = e_k$ as the initial conditions to construct $\Phi(t)$. We can start with any n-linearly independent solutions $\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n$ of the ODE $\dot{x} = A(t)x$. If we define $\widetilde{\Phi}(t) = [\tilde{v}_1\ \tilde{v}_2\ \cdots \tilde{v}_n]$, then $\Phi(t) = \widetilde{\Phi}(t)\widetilde{\Phi}(0)^{-1}$ is the desired fundamental matrix.

Remark. The fundamental matrix $\Phi(t,s)$ exists for general linear ODEs $\dot{x} = Ax$, where A is any general matrix (no need to be periodic).

Now we can see how the periodicity of the coefficient matrix A appears in the solution. In general, for non-autonomous ODEs (as the ones with periodic coefficients considered in this section), if $v(t)$ is a solution, $v(t+s)$ does not have to be a solution. But for periodic solution, if we differentiate both sides of $\tilde{v}(t) = v(t+T)$, then

$$\frac{d}{dt}\tilde{v}(t) = \dot{v}(t+T) = A(t+T)v(t+T) = A(t)\tilde{v}(t).$$

That is $\tilde{v}(t) = v(t+T)$ is also a solution. Then from the facts that

$$\tilde{v}(t) = \Phi(t)\tilde{v}(0) = \Phi(t)v(T) = \Phi(t)\Phi(T)v(0)$$

and $v(t+T) = \Phi(t+T)v(0)$ for any $v(0)$, we get $\Phi(t+T) = \Phi(t)B$ and $\Phi(nT) = B^n$ for any integer n, where $B \equiv \Phi(T)$ is called the *monodromy matrix* of the system. Similar to the one dimensional case, if there is a constant matrix H such $B = \exp(TH)$, then $\Phi(t)\exp(-tH)$ is periodic, or

$$\Phi(t) = P(t)\exp(tH),$$

the fundamental matrix $\Phi(t)$ is the product of a periodic matrix $P(t)$ and a matrix exponential $\exp(tH)$. The stability of periodic solutions is reduced to the eigenvalues of the monodromy matrix $B = \Phi(T) = \exp(TH)$: if all eigenvalues have modulus less than unit, the periodic solutions are locally stable; otherwise if there is one eigenvalue with modulus greater than unit, then the periodic solutions are unstable.

Example 4.9 (Another linear ODEs with periodic coefficients)**.** If

$$A(t) = \begin{pmatrix} -1+\frac{1}{2}\cos^2 t & 1-\frac{1}{2}\cos t\sin t \\ -1-\frac{1}{2}\cos t\sin t & -1+\frac{1}{2}\sin^2 t \end{pmatrix}$$

with period π or

$$\begin{aligned} \dot{u} &= (-1+\tfrac{1}{2}\cos^2 t)u + (1-\tfrac{1}{2}\cos t\sin t)v \\ \dot{v} &= (-1-\tfrac{1}{2}\cos t\sin t)u + (-1+\tfrac{1}{2}\sin^2 t)v \end{aligned}$$

Then

$$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{-\frac{1}{2}t}, \qquad \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}$$

are solutions with initial conditions e_1 and e_2. Therefore

$$\Phi(t) = \begin{pmatrix} e^{-\frac{t}{2}}\cos t & e^{-t}\sin t \\ -e^{-\frac{t}{2}}\sin t & e^{-t}\cos t \end{pmatrix}$$

and

$$B = \Phi(\pi) = \begin{pmatrix} e^{-\frac{1}{2}\pi} & 0 \\ 0 & e^{-\pi} \end{pmatrix}$$

and hence the origin is stable.

For linear ODEs with periodic coefficients from linearisation, more information is available about the eigenvalues.

Theorem 4.3 (Special value w.r.t perturbed periodic solutions)**.** *Let $\phi(t)$ be a periodic solution of the autonomous system $\dot{x} = f(x)$, and $\dot{v} = A(t)v$ with $A(t) = Df(\phi(t))$ is the linearisation around $\phi(t)$. Then the monodromy matrix B corresponding to $A(t)$ always have eigenvalue 1.*

Proof. Let $v = \dot{\phi}$, then taking derivative of the equations for ϕ, $v = f(\phi(t))$, we have

$$\dot{v} = Df(\phi(t))\dot{\phi}(t) = A(t)v.$$

That is the derivative $v = \dot{\phi}$ satisfies the linearised ODEs and is periodic with period T (the same period as ϕ). Therefore,

$$v(0) = v(T) = \Phi(T)v(0) = Bv(0),$$

and 1 is an eigenvalue of B with eigenvector $v(0) = \dot{\phi}(0)$. □

Another result is to generalise the solution $x(t) = x_0 \exp(\int_0^T a(s)ds)$ of the scalar ODE $\dot{x} = a(t)x$ into higher dimension.

Theorem 4.4 (Evolution of the determinant)**.** *If $\Phi(t)$ is a non-singular matrix that satisfies the system of ODEs $\dot{x} = A(t)x$ (the matrix A does not have to be periodic and $\Phi(0)$ does not have to be the identity matrix), then*

$$\det\Phi(t) = \exp\left(\int_s^t \mathrm{tr}A(s)ds\right)\det\Phi(s).$$

Sketched proof. We can actually show the equivalent versions

$$\frac{d}{dt}\det\Phi(t) = \mathrm{tr}A(t)\det\Phi(t).$$

Without loss of generality, we can assume $s = 0$ and $\Phi(0) = I$. Then $\Phi(t) = I + tA(0) + O(t^2)$ when t is small. Then

$$\det\Phi(t) = \det\big(I + tA(0) + O(t^2)\big) = 1 + t\mathrm{tr}A(0) + O(t^2).$$

Therefore, $\frac{d}{dt}\det\Phi(t)\big|_{t=0} = \mathrm{tr}A(0)$. □

Remark. This theorem is a more general fact: if the points evolve under the ODE $\dot{x} = f(x,t)$, then the rate of change of the local volume element is $\mathrm{div}f(x,t)$, the divergence of the vector field $f(x,t)$. The volume (or area in two dimension) does not change if and only if the divergence of the vector field is zero. In the special case of linear ODEs with $f(x,t) = A(t)x$, we recover the above result since $\mathrm{div}f(x,t) = \mathrm{tr}A(t)$.

Now we can look at the stability. The eigenvalues of the monodromy matrix are denoted as $\rho_1, \rho_2, \cdots, \rho_n$, also called *characteristic multipliers*. Their logarithms divided by T are called *characteristic exponents*, i.e. $\rho_k = \exp(\mu_k T)$. Therefore

$$\rho_1\rho_2\cdots\rho_n = \exp(\mu_1 T)\exp(\mu_2 T)\cdots\exp(\mu_n T) = \det(B) = \exp\left(\int_0^T \mathrm{tr}A(s)ds\right).$$

If the ODEs $\dot{v} = A(t)v$ is two dimensional, and are derived from periodic solutions of

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2).$$

Then we have one characteristic multiplier $\rho_1 = 1$ and the other one

$$\rho_2 = \exp\left(\int_0^T \mathrm{tr}A(s)ds\right) = \exp\left(\int_0^T \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) ds\right).$$

Therefore, the stability is determined by the *sign* of the integral of the divergence of the vector field (f_1, f_2) along the periodic solution.

Example 4.10. Consider the ODEs

$$\dot{x} = x - y - x(x^2 + y^2), \qquad \dot{y} = x + y - y(x^2 + y^2).$$

In polar coordinates $x(t) = r(t)\cos\theta(t), y(t) = r(t)\sin\theta(t)$, the equations become

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

There is a periodic solution $r(t) = 1$ with period $T = 2\pi$. The stability of this periodic solution is easy in the polar coordinates, or using the criteria above. We have

$$A = \frac{\partial}{\partial x}\left(x - y - x(x^2 + y^2)\right) + \frac{\partial}{\partial y}\left(x + y - y(x^2 + y^2)\right) = 2 - 4(x^2 + y^2).$$

Therefore, on the periodic orbits $r = 1$ (or $x^2 + y^2 = 1$), $A(t)$ is the constant $-2 < 0$ or $\rho_2 = \exp(-2T) = \exp(-4\pi) < 1$. The periodic solution is stable.

4.3 Return maps

Suppose Γ is a periodic orbit in phase space. Then pick some point $p \in \Gamma$ and let Σ be an $(n-1)$-dimensional surface containing p which is transverse to the flow (so the flow always crosses Σ in the same direction). Then the flow defines a map from this surface back to itself close to the periodic orbit. More specifically, let U be a small neighbourhood of p and $V = U \cap \Sigma$, then define the return map (or Poincaré map) $R : V \to \Sigma$ by $R(x) = y$ if

$$y = \phi(x, \tau) \in \Sigma \quad \text{and} \quad \phi(x, t) \notin \Sigma, \quad 0 < t < \tau.$$

Note that by definition $R(p) = p$ so p is a fixed point of the return map.

Example 4.11. Consider the system (in polar coordinates)

$$\dot{r} = r(a^2 - r^2), \quad \dot{\theta} = \omega,$$

for some constant $a > 0$.

There is a periodic orbit with $r = a$. Let Σ be the positive x-axis, i.e. $\theta = 0 \pmod{2\pi}$, and start at $(x_0, y_0) = (a + \varepsilon, 0)$ or $(r_0, \theta_0) = (a + \epsilon, 0)$. The $\theta = \omega t$ and

$$\begin{aligned} t = \int_{r_0}^r \frac{dr}{r(a^2 - r^2)} = \frac{1}{2}\int_{r_0}^r \frac{dr^2}{r^2(a^2 - r^2)} &= \frac{1}{2a^2}\int_{r_0}^r \left[\frac{1}{r^2} - \frac{1}{r^2 - a^2}\right] dr^2 \\ &= \frac{1}{2a^2}\ln\left(\frac{r^2}{r^2 - a^2}\frac{r_0^2 - a^2}{r_0^2}\right). \end{aligned}$$

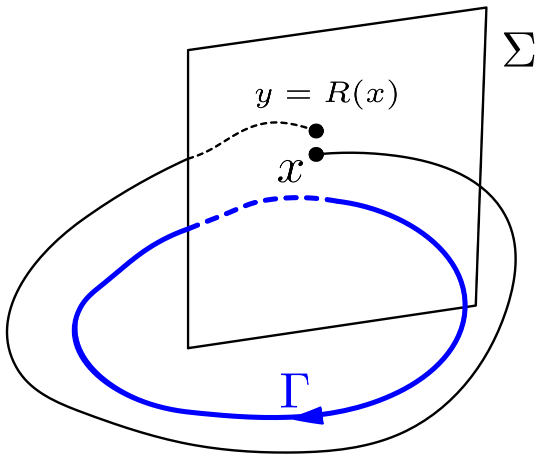

Figure 4.4: Return map characterising the periodic orbit Γ.

Eliminating t using $\theta = \omega t$, we get r as a function of θ:

$$r = a\left(1 + \frac{a^2 - r_0^2}{r_0^2} e^{-2a^2 t}\right)^{-1/2} = a\left(1 + \frac{a^2 - r_0^2}{r_0^2} e^{-2a^2\theta/\omega}\right)^{-1/2}.$$

The return map is given at $\theta = 2\pi, 4\pi, \cdots$, or equivalently $t = 2\pi/\omega, 4\pi/\omega, \cdots$ (we can set $T = 2\pi/\omega$), and correspondingly

$$r(T) = a\left(1 + \frac{a^2 - r_0^2}{r_0^2} e^{-2a^2 T}\right)^{-1/2} = \left(1 + \frac{a^2 - r_0^2}{r_0^2} e^{-4a^2\pi/\omega}\right)^{-1/2}.$$

For any $r_0 > 0$, it is easy to see that $r(nT) \to a$ as n goes to infinity. That is the return map eventually converges to $r = a$, the circle.

Remark. It is not usually possible to solve analytically in this way, but could be easy to construct numerically. For this reason, this section about return maps will not be assessed in the coursework or in the final exam.