# 3 Linearisation and equilibria

In this section, we will study mainly the properties of linear systems around fixed points, as a first step towards the understanding of more complicated behaviours of nonlinear systems. The local, linear part can be obtained from the full nonlinear counterpart via Taylor expansion around the fixed points.

## 3.1 Taylor's theorem

Suppose $x^* \in \mathbb{R}^n$ is a stationary point of $\dot{x} = f(x)$, that is $f(x^*) = 0$. If $z = x - x^*$ is small, we can use Taylor's Theorem to expand $f(x)$ around $x^*$, that is,

$$\dot{x} = \dot{z} = f(x) = f(x^* + z) = \cancel{f(x^*)} + Df(x^*)z + O(|z|^2) = Df(x^*)z + O(|z|^2).$$

Here $Df(x^*)$ is the $n \times n$ *Jacobian matrix* with entries $[Df(x)]_{ij} = \frac{\partial f_i}{\partial x_j}(x)$. The 'big-O' notation means that if a function $g(z) = O(|z|^2)$ then $\frac{|g(z)|}{|z|^2} < C$, for some $C < \infty$, on a neighbourhood of $z = 0$. If $z$ is small then we can hope to ignore the small $O(|z|^2)$ terms and consider the linearisation about $x^*$: $\dot{z} = Az$ with $A = Df(x^*)$ or $A_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$ .

**Example 3.1.** Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} \frac{5}{2}x - \frac{1}{2}y + 2x^2 + \frac{1}{2}y^2 \\ -x + 2y + 4xy \end{pmatrix}, \tag{3.1}$$

for which $(0,0)$ is a stationary point. Since

$$Df(x,y) = \begin{pmatrix} \frac{5}{2} + 4x & -\frac{1}{2} + y \\ -1 + 4y & 2 + 4x \end{pmatrix},$$

the system can be approximated by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad A = Df(0,0) = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}.$$

which could have been read off directly from the linear part of the equation (3.1).

**Key questions in the next few subsections**

1. How can we characterize solutions of linear equations $\dot{x} = Ax$?
2. (Harder) How/when does information about the linearisation provide useful local information about the original (nonlinear) problem?

**Example 3.2.** Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}. \tag{3.2}$$

We start with the stationary points, by looking for zeros of the right hand side of the system: the second equation implies that $x = 0$; substituting it back into the first equation, we get $y = -1$. Therefore, the only stationary point is $(0, -1)$.

From $f(x,y) = \begin{pmatrix} 1-ax^2+y \\ bx \end{pmatrix}$, $Df(x,y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$ and $Df(0,-1) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$. The linearisation in coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$ is $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ or

$$\dot{u} = v, \qquad \dot{v} = bu.$$

This linsear system can be solved by eliminating $v$ (or alternatively using matrix exponential): $\ddot{u} = \dot{v} = bu$, or $\ddot{u} - bu = 0$. Using elementary methods in ODEs, if $b > 0$ the solutions are

$$u = Ae^{\sqrt{b}t} + Be^{-\sqrt{b}t} \quad \text{with} \quad v = \dot{u} = \sqrt{b}(Ae^{\sqrt{b}t} - Be^{-\sqrt{b}t}),$$

for constants $A$ and $B$ determined from the initial condition. Most solutions are unbounded (with general $A$ and $B$). But the special solution with $A = 0$ converges to the origin.

If $b < 0$ then

$$u = A\cos\sqrt{|b|}t + B\sin\sqrt{|b|}t \quad \text{and} \quad v = \sqrt{|b|}\big(-A\sin\sqrt{|b|}t + B\cos\sqrt{|b|}t\big)$$

i.e. solutions of the linearisation oscillate in time.

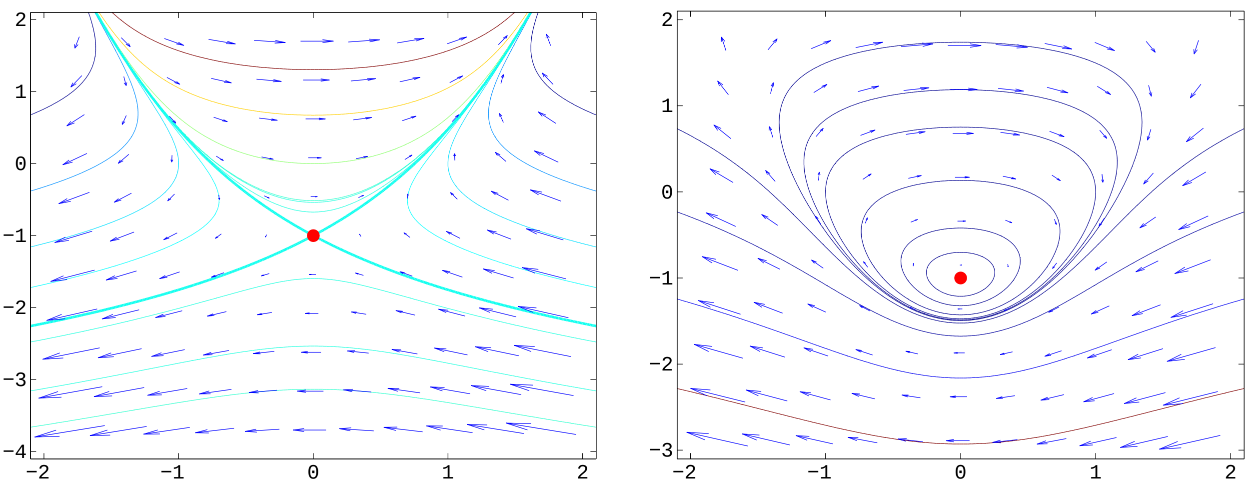



Figure 3.1: The phase portrait for two systems: (left figure) $\dot{x} = 1 - x^2 + y, \dot{y} = x$ and (right figure) $\dot{x} = 1 - x^2 + y, \dot{y} = -x$, with common fixed point $(0, -1)$.

**Question:** When does this linear analysis give accurate information about the behaviour of the full nonlinear problem? It will turn out that the behaviour if $b > 0$ is a good sense of the general behaviour (locally) whilst this may not be the case if $b < 0$. For instance, the trajectories of the system $\dot{x} = 1 - x^2 + y, \dot{y} = -x - x^2$ around the fixed point $(0, -1)$ are spirals.

For the above system $\dot{x} = 1 - ax^2 + y, \dot{y} = bx$, you can show that

$$V(x,y) = (2a^2x^2 - 2a + b - 2ay)\exp(2ay/b)$$

is conserved under the full system (3.2), and the trajectories are governed by $V(x,y) = C$ for different constants $C$.

## 3.2 Linear systems

Suppose $x(t)$ satisfies the linear ODEs $\dot{x} = Ax$, where $A$ is a constant $n \times n$ matrix and $x \in \mathbb{R}^n$. If $A$ has distinct eigenvalues $\lambda_i$ with corresponding eigenvectors $e_i$, then the general solution is a superposition of the eigenmodes:

$$x(t) = \sum C_k e^{\lambda_k t} e_k$$

where the $C_k$ are constants determined from the initial condition $x(0) = \sum C_k e_k$. This shows that eigenvalues and eigenvectors of $A$ will be crucial to the understanding of the dynamics.

The eigen-pairs are also closely related to a particular coordinate transformation that simplifies the dynamics: $x = [e_1\ e_2\ \cdots\ e_n]y$, or $y = Ux$ with $U = [e_1\ e_2\ \cdots\ e_n]^{-1}$, the inverse of the matrix formed by the eigenvectors. The the ODE for $y$ becomes

$$\dot{y} = U\dot{x} = UAx = UAU^{-1}y,$$

i.e. $y$ satisfies a linear ODE $\dot{y} = UAU^{-1}y$.

The above choice of $U = [e_1\ e_2\ \cdots\ e_n]^{-1}$ is particular in that $UAU{-}1$ is *diagonal*, and the system $\dot{y} = UAU{-}1y$ is essential $n$ decoupled ODEs:

$$\dot{y}_1 = \lambda_1 y_1, \quad \dot{y}_2 = \lambda_2 y_2, \quad \cdots \quad \dot{y}_n = \lambda_n y_n.$$

With this 'natural' choice of transformation $y = Ux$, the resulting system for $y$ is called the *normal form* (depending only on the eigenvalues of $A$). We will work in the plane $\mathbb{R}^2$ with real matrix $A$, though extension to $\mathbb{R}^n$ is not hard.

**a)** **eigenvalues real and distinct:** Suppose the eigenvalues $\lambda_1$ and $\lambda_2$ of $A$ are real and distinct, then their corresponding eigenvectors $e_1, e_2$ (assumed to be column vectors) are real and linearly independent. With the matrix $U = [e_1, e_2]^{-1}$, we get

$$AU^{-1} = A[e_1\ e_2] = [Ae_1\ Ae_2] = [\lambda_1 e_1\ \lambda_2 e_2] = [e_1\ e_2]\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix} = U^{-1}\mathrm{diag}(\lambda_1, \lambda_2).$$

Left multiplying both sides with $U$, we have $UAU^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$ as expected.

As we shall see shortly, this transformation into normal form also makes it easier to understand the structure of the solutions, which depends on the signs of $\lambda_1$ and $\lambda_2$.

**a i)** $\lambda_1 < \lambda_2 < 0$**: stable node** In this case the linearisation in the normal form coordinates $y^t = (u, v)$ is

$$\dot{u} = \lambda_1 u, \quad \dot{v} = \lambda_2 v$$

with solutions

$$u = u_0 e^{\lambda_1 t}, \quad v = v_0 e^{\lambda_2 t}.$$

Thus $(u, v) \to (0, 0)$ as $t \to \infty$ and both coordinate axis ($u = 0$ and $v = 0$) are invariant.

Moreover, if $u_0, v_0 \neq 0$ (i.e. off the coordinate axes)

$$\frac{u}{u_0} = e^{\lambda_1 t} = (e^{\lambda_2 t})^{\frac{\lambda_1}{\lambda_2}}; \qquad \frac{v}{v_0} = e^{\lambda_2 t}$$

and so

$$\frac{u}{u_0} = (\frac{v}{v_0})^{\frac{\lambda_1}{\lambda_2}}, \qquad \frac{\lambda_1}{\lambda_2} > 1, \tag{3.3}$$

or equivalently $uv^{-\lambda_1/\lambda_2}$ is a constant for points on the same trajectory. These are generalized parabolas, tangential to the $v$-axis at $(u,v) = (0,0)$.

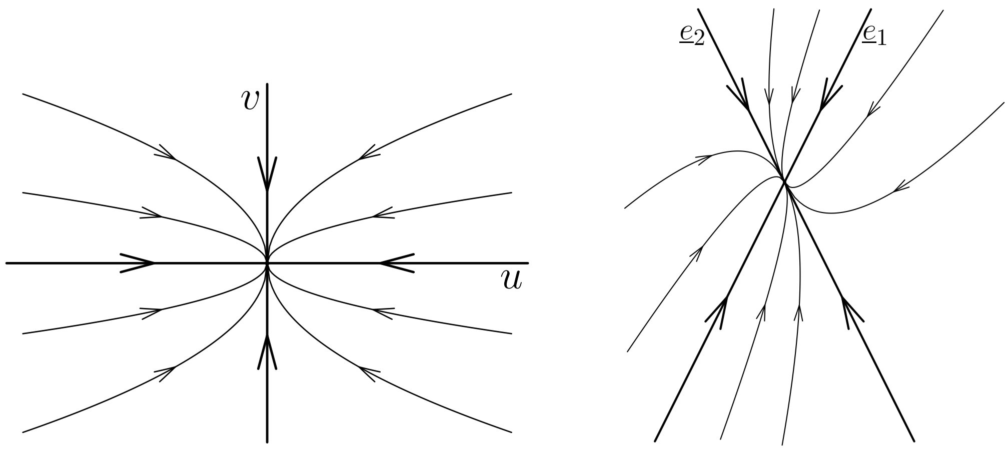



Figure 3.2: Stable node in transformed $(u,v)$-coordinates and in the original $(x_1,x_2)$-coordinates.

This is called a *stable node*. In the original coordinates the $u$-axis corresponds to $\underline{e}_1$ and the $v$-axis to the $\underline{e}_2$ direction (you can see this from the transformation $y = Ux$), so the phase portrait is as shown in Figure 3.2.

Thus in the original coordinates, lines corresponding to eigenvectors are invariant. Moreover almost all trajectories are tangential to $\underline{e}_2$ at $(0,0)$, i.e. tangential to eigenvector of eigenvalue with smallest modulus.

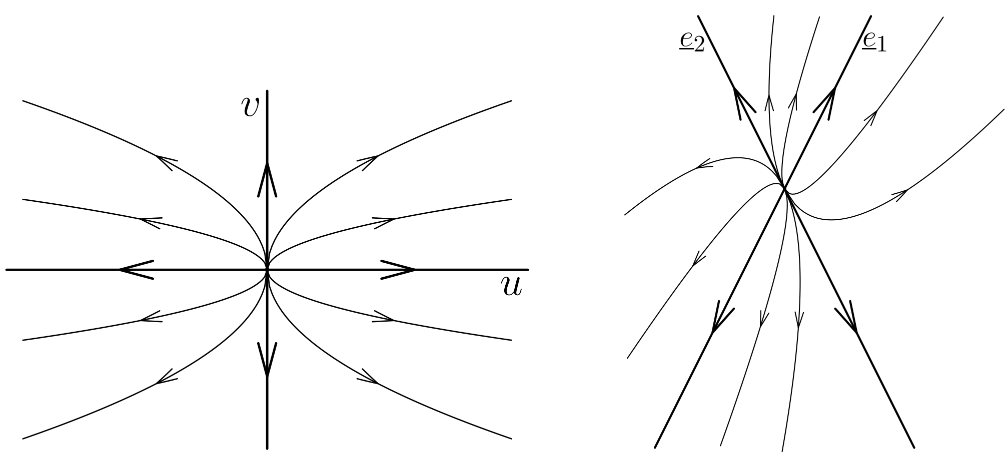



Figure 3.3: Unstable node in transformed $(u,v)$-coordinates and in the original $(x_1,x_2)$-coordinates.

**a ii) $\lambda_1 > \lambda_2 > 0$: unstable node** The phase portrait can be obtained using the same techniques as in the previous section. Indeed the manipulations are the same and the generalized parabola is also the same as changing the signs of both eigenvalues does not change the sign of their ratio. Another way of seeing the direct correspondence with the previous case is by reversing time. Set $\tau = -t$ so

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = -\frac{\mathrm{d}}{\mathrm{d}\tau}$$

and so if

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \lambda_1 u, \quad \frac{\mathrm{d}}{\mathrm{d}t}v = \lambda_2 v$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u = -\lambda_1 u, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}v = -\lambda_2 v$$

which is the same as in case ai). Thus all we need to do is to change the direction of time, i.e. the arrows on the phase portraits to get the new phase portrait. This is called an *unstable node*, as shown in Figure 3.3.

**a iii)** $\lambda_1 < 0 < \lambda_2$**: saddle** The analysis is as before but now $\frac{u}{u_0} = (\frac{v}{v_0})^{\frac{\lambda_1}{\lambda_2}}$ is a generalized hyperbola as $\lambda_1/\lambda_2 < 0$, as shown in Figure 3.4.

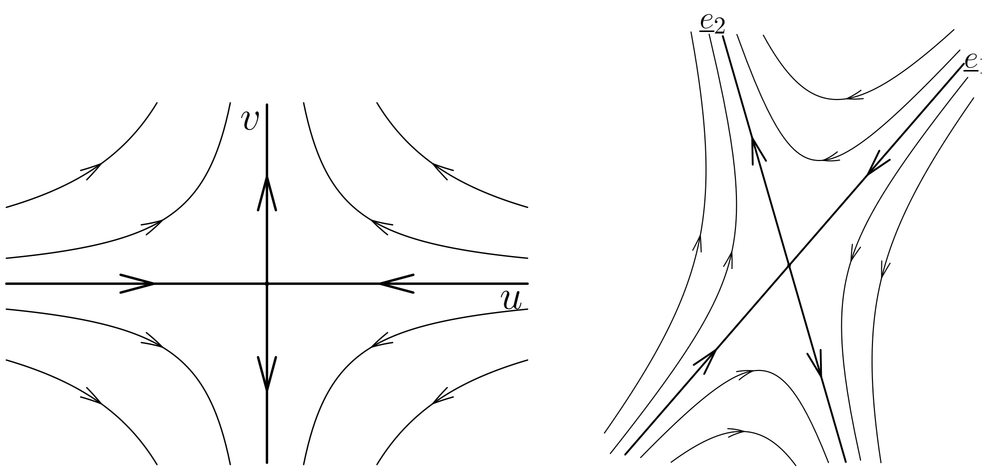



Figure 3.4: Saddle node in transformed $(u, v)$-coordinates and in the original $(x_1, x_2)$-coordinates.

**b) Complex conjugate eigenvalues** $\rho \pm i\omega, \omega \neq 0$**.** The eigenvectors $z_\pm$ are *complex*, and satisfy

$$Az_\pm = (\rho \pm i\omega)z_\pm. \tag{3.4}$$

But we prefer to work with *real* quantities, and the first step is to take the real and imaginary parts of both sides of (3.4) (only with $z_+$),

$$A(\mathrm{Re}z_+ + i\mathrm{Im}z_+) = (\rho + i\omega)(\mathrm{Re}z_+ + i\mathrm{Im}z_+),$$

or equivalently

$$A\mathrm{Re}z_+ = \rho\mathrm{Re}z_+ - \omega\mathrm{Im}z_+, \qquad A\mathrm{Im}z_+ = \rho\mathrm{Im}z_+ + \omega\mathrm{Re}z_+.$$

To proceed, we take real and imaginary parts of the above eigenvector $z_\pm$ (remember that the real parts of $z_\pm$ are the same, and the imaginary parts only differ in their signs), forming the matrix

$$U = [\mathrm{Re}z_+ \ , \ \mathrm{Im}z_+]^{-1} \qquad \text{or} \qquad U^{-1} = [\mathrm{Re}z_+ \ , \ \mathrm{Im}z_+] \, .$$

Then

$$\begin{aligned}
AU^{-1} &= [A\ \mathrm{Re}z_+,\ A\ \mathrm{Im}z_+] \\
&= [\mathrm{Re}(Az_+)\ ,\ \mathrm{Im}(Az_+)]\,, \qquad\qquad (\text{ as } A \text{ is real}) \\
&= [\rho\mathrm{Re}(z_+) - \omega\mathrm{Im}(z_+),\ \rho\mathrm{Im}(z_+) + \omega\mathrm{Im}(z_+)] \\
&= [\mathrm{Re}z_+, \mathrm{Im}z_-]\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}.
\end{aligned}$$

Thus we end up with

$$\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix} = UAU^{-1},$$

where $\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}$ is the *complex normal form.*

In the new coordinates $y = (u,v)^t = Ux$, the system becomes $\dot{u} = \rho u + \omega v, \dot{v} = -\omega u + \rho v$. It is much easier to look at this system in the polar coordinates $u = r\cos\theta, v = r\sin\theta$. Differentiating this new transform gives

$$\begin{aligned}
\dot{u} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta = \rho r\cos\theta + \omega r\sin\theta \\
\dot{v} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -\omega r\cos\theta + \rho r\sin\theta.
\end{aligned}$$

To eliminate $\dot{\theta}$ to obtain an equation for $\dot{r}$, multiply the first of these by $\cos\theta$ and the second by $\sin\theta$ and add to get

$$\dot{r} = \rho r, \quad \text{i.e.} \quad r = r_0 e^{\rho t}. \tag{3.5}$$

Similarly to get the equation for $\dot{\theta}$, multiply the first by $\sin\theta$ and the second by $\cos\theta$ and take the difference:

$$\dot{\theta} = -\omega \quad \text{i.e.} \quad \theta = \theta_0 - \omega t,$$

which represents a clockwise rotation at constant angular velocity if $\omega > 0$. Using this to eliminate $t$ from the equation for $r$ shows that trajectories lie on spiral $r = r_0 e^{\rho(\theta-\theta_0)/w}$.

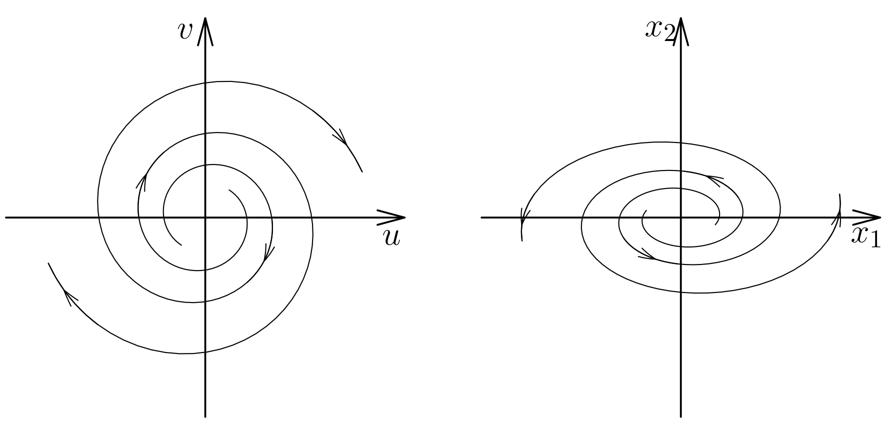



Figure 3.5: Unstable focus in transformed $(u,v)$-coordinates and in the original $(x_1,x_2)$-coordinates.

**b i) $\rho > 0$: unstable focus.** In this case (3.5) shows that solutions grow with time so trajectories spiral out of the origin. This is called a *unstable focus* clockwise if $\omega > 0$ (counter-clockwise if $\omega < 0$). In the original coordinates, the phase portrait is a distorted spiral. To determine the direction of spiralling, we can consider the sign of $\dot{x}_2$ on a horizontal line (where $x_2 = 0$) through the stationary point or the sign of $\dot{x}_1$ on a vertical line through the stationary point. If more detail is required the nullclines (see c(ii) below) indicate where solutions are flat or vertical as they move around the stationary point. This is called an *unstable focus* (see Figure 3.5).

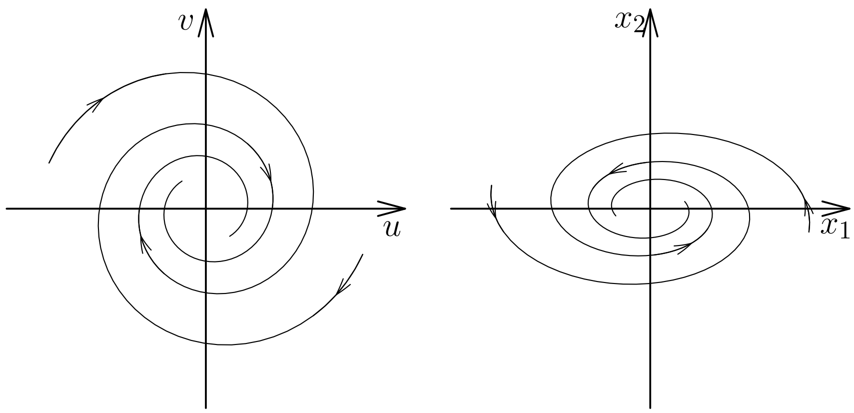



Figure 3.6: Stable focus in transformed $(u, v)$-coordinates and in the original $(x_1, x_2)$-coordinates.

**b ii) $\rho < 0$ : stable focus** In this case the $\theta$ behaviour is the same as in the previous case but the radial velocity is towards zero. Solutions tend to the origin spiralling clockwise (if $\omega > 0$) as shown in Figure 3.6. In the original coordinates the solutions spiral inwards, with the direction given by consideration of the sign of $\dot{x}_2$ (or $\dot{x}_1$) on the horizontal line (resp. vertical line) through the stationary point. If more detail is required the nullclines (see c(ii) below) indicate where solutions are flat or vertical as they move around the stationary point. This is called a *stable focus*.

**b iii) $\rho = 0$: centre** If $\rho = 0$ then $\dot{r} = 0$ and so $r$ is constant – solutions lie on circles in the transformed $(u, v)$-coordinates with $\theta$ changing at a constant rate, i.e. if $r_0 \neq 0$ then solutions are periodic with period $\frac{2\pi}{|\omega|}$. This is called a *centre*, see Figure 3.7.

Clearly if nonlinear terms are added then $\dot{r}$ may no longer vanish, so we do not expect this type of behaviour to persist in typical nonlinear systems.

**c) Repeated real roots $\lambda \neq 0$.** If the characteristic equation has two repeated roots $\lambda = \lambda_1 = \lambda_2$, then by Cayley-Hamiltonian Theorem (a matrix satisfies its own characteristic equation) $(A - \lambda I)^2 = 0$. Depending on the number of eigenvectors to the equation $(A - \lambda I)e = 0$, we have two cases (the equivalent two cases are $A = \lambda I$ and $A \neq \lambda I$).

**c i) Repeated real roots $\lambda \neq 0$: star.** Suppose there are two (linearly independent) eigenvectors $e_1$ and $e_2$ to the equation $(A - \lambda I)e = 0$, then

$$A[e_1, e_2] = [\lambda e_1, \lambda e_2] = \lambda[e_1, e_2].$$

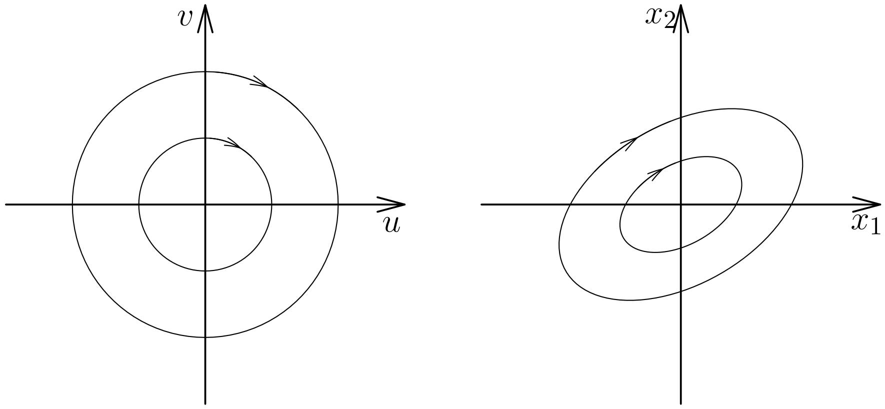
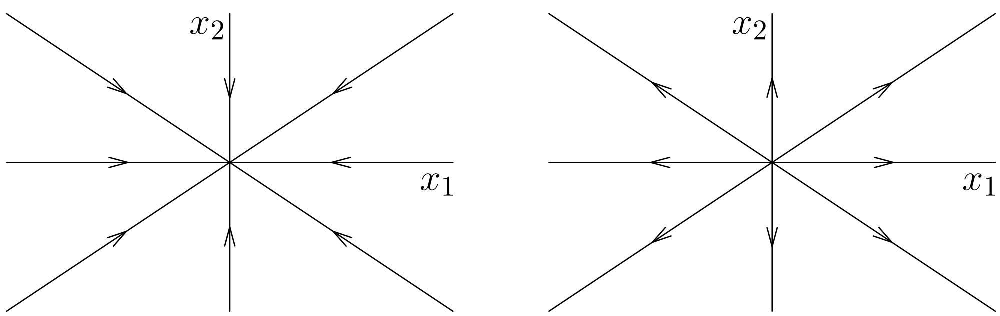



Figure 3.7: Centre in transformed $(u, v)$-coordinates and in the original $(x_1, x_2)$-coordinates.

Since $[e_1, e_2]$ is non-singular, we can right multiply $[e_1, e_2]^{-1}$ to the previous equation to get

$$A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

i.e. $A$ is diagonal in *any* basis! Then using (3.3) with $\lambda_1 = \lambda_2$, $\frac{x}{x_0} = \frac{y}{y_0}$ and so solutions lie on straight lines through the origin as shown in Figure 3.8. This is called a *stable star* if $\lambda < 0$ (so solutions tend to the origin) and an *unstable star* if $\lambda > 0$ (so solutions grow).




Stable star ($\lambda < 0$) Unstable star ($\lambda > 0$)

Figure 3.8: Stable stars and unstable stars

**c ii): repeated roots $\lambda \neq 0$: degenerate node.** Suppose that there is only one eigenvalue $e_1$ to the eigenvalue $\lambda$ (although it is repeated), then we can find another vector $e_2$ such that $(A - \lambda I)e_2 = e_1$, or $Ae_2 = \lambda e_2 + e_1$. Let $U^{-1} = [e_1; e_2]$, the matrix with columns are the two vectors defined above. Then

$$AU^{-1} = [Ae_1, Ae_2] = [\lambda e_1, \lambda e_2 + e_1] = [e_1, e_2] \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Hence we get the

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = UAU^{-1},$$

where the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is the *normal form* for this case of repeated roots.

In the transformed coordinates $(u, v)$ defined as above,

$$\dot{u} = \lambda u + v, \quad \dot{v} = \lambda v. \tag{3.6}$$

The second equation is easily solved to give $v = v_0 e^{\lambda t}$. Substituting this into the first equation gives $\dot{u} = \lambda u + v_0 e^{\lambda t}$. Using the integrating factor $e^{-\lambda t}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(ue^{-\lambda t}) = v_0.$$

The integration of both sides lead to $ue^{-\lambda t} - u_0 = v_0 t$ or $u = u_0 e^{\lambda t} + v_0 t e^{\lambda t}$. It is hard to analyse solutions directly from (3.6). The phase portrait is given in Figure 3.9. They divide phase space into four regions according to the different combinations of the signs of $\dot{u}$ and $\dot{v}$. Bu considering the behaviour in each of these regions, we arrive at the phase portrait sketched. Of course, a rigorous justification takes more work, but this is enough to give a basic idea of the behaviour. The phase portrait of Figure 3.9 is an *unstable degenerate node*. For the case $\lambda < 0$ (a *stable degenerate node*), although the solution eventually converges to the origin, it may take a long excursion to finally move towards it.

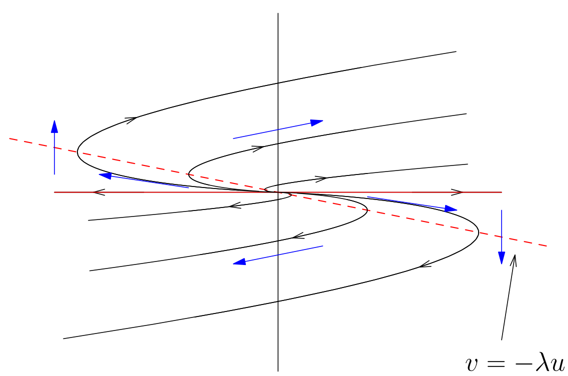



Figure 3.9: Phase portrait for $\dot{u} = \lambda u + v, \dot{v} = \lambda v$ with degenerate node ($\lambda > 0$).

## Direction of rotation for foci and centres

The sign of $\omega$ in the normal form can always be chosen to be positive, but this might use a transformation that reverses the orientation of the plane, i.e. counter-clockwise rotation can be transformed into clockwise rotation. To determine the actual direction in an example either calculate the nullclines and the direction of the flow across the nullclines, or (and this is often easier) consider the direction of the flow on coordinate axes.

**Example 3.3.** Suppose that

$$\dot{x} = -x - 4y, \quad \dot{y} = x - y$$

so the Jacobian matrix is

$$\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

with characteristic equation $(s+1)^2 + 4 = 0$ so the eigenvalues are $-1 \pm i2$. The origin is therefore a stable node, but in which way does it rotate?

Set $x = 0$ (the $y$-axis) and consider how solutions move across this line. On $x = 0$ $\dot{x} = -4y$ and so if $y > 0$ then the motion is from right to left (as $\dot{x} < 0$) and if $y < 0$ (so $\dot{x} > 0$) the motion is from left to right. Thus the motion is counter-clockwise.

It is often useful to indicate this on a diagram to make sure the figure is drawn appropriately.

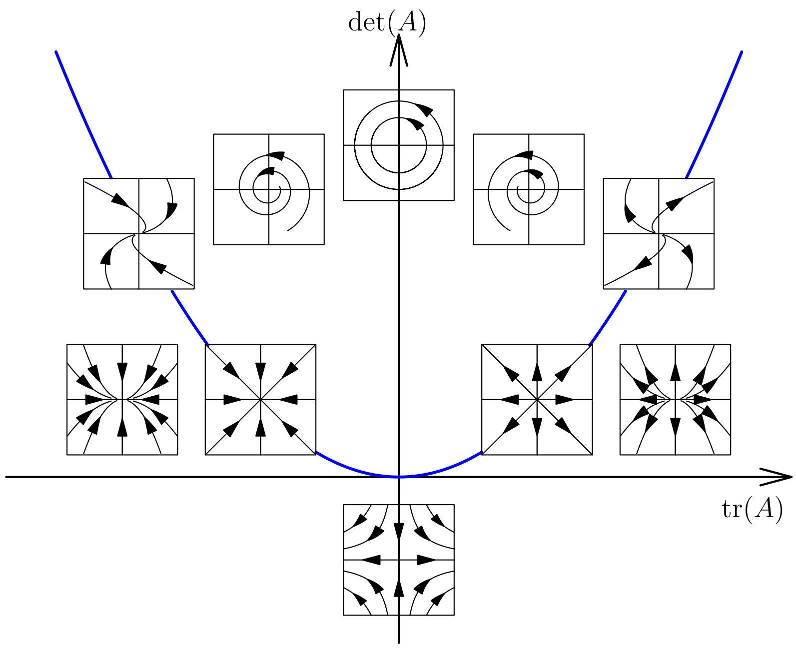



Figure 3.10: Different behaviours of the system $\dot{x} = Ax$, depending on the determinant and the trace of $A$.

**Summary on the relation between the signs of eigenvalues and the behaviour of the underlying linear system**

Given $\dot{x} = Ax$, $x \in \mathbb{R}^2$. Find the eigenvalues of $A$ to characterise all the behaviours of the solution summarised as below.

**real, distinct:** *both positive*: unstable node, almost all trajectories tangential to eigenvector of eigenvalue with smaller modulus at $(0, 0)$

*both negative*: unstable node, almost all trajectories tangential to eigenvector of eigenvalue with smaller modulus at $(0, 0)$

*one negative one positive*: saddle with both eigenvectors invariant

**complex conjugate pair:** *positive real parts*: unstable focus

*negative real parts*: stable focus

*zero real parts*: centre

**other special cases:** *equal (real) eigenvalues with two eigenvectors:* star

*equal (real) eigenvalues with only one eigenvectors:* degenerate node otherwise, almost all trajectories tangential to eigenvector at $(0,0)$.

Since the behaviours of the solutions, or equivalently of the roots depend only on the determinant and the trace of $A$, they can be equally summarised as in Figure 3.10, where the parabola is the curve $4\det A = (\mathrm{tr}A)^2$.

## 3.3 Planar ODEs

Recall that if $x = a$ is a stationary point then we use $x = a + y$, $|y| \ll 1$ to change the coordinate to $y$, such that

$$\dot{y} = Df(a)y + O(|y|^2)$$

ad linearisation is $\dot{y} = Ay$, $A = Df(a)$ (Jacobian matrix of partial derivatives).

It turns out that nodes, foci and degenerate nodes retain their basic properties under small nonlinear perturbations, so this makes it possible to obtain approximate phase portraits for some systems (ignoring periodic orbits for the moment).

**Example 3.4** (ODEs for competitive populations)**.** Imagine a colony of rabbits ($r$) and sheep ($s$) with $r, s \geq 0$ denoting the population size in normalized coordinates so that one unit represents many animals and we are justified in approximating the population size as a continuous variable. A model of the birth/death rates is

$$\begin{aligned} \dot{r} &= r(3 - r - s), \qquad \text{rabbits} \\ \dot{s} &= s(4 - 2r - s), \qquad \text{sheep} \end{aligned}$$

where $s, r \geq 0$.

We can sketch the phase portrait in three stages: first find the stationary points, then determine their types and the local phase portrait assuming the linear approximation is valid, and then put this information together to create a consistent global phase portrait.

**Stationary Points:** $\dot{r} = 0$ if $r = 0$ or $r + s = 3$ whilst $\dot{s} = 0$ if $s = 0$ or $2r + s = 4$. Hence the stationary points are

$$\begin{aligned} r &= 0, \quad s = 0, \qquad (0,0) \\ r &= 0, \quad s = 4, \qquad (0,4) \\ s &= 0, \quad r = 3, \qquad (3,0) \end{aligned}$$

together with the solution of the simultaneous equations

$$r + s = 3, \qquad 2r + s = 4$$

if they exist. Solving gives a fourth stationary point, $r = 1$ and $s = 2$, i.e. $(1,2)$.

Note that the $r$-axis and the $s$-axis are invariant.

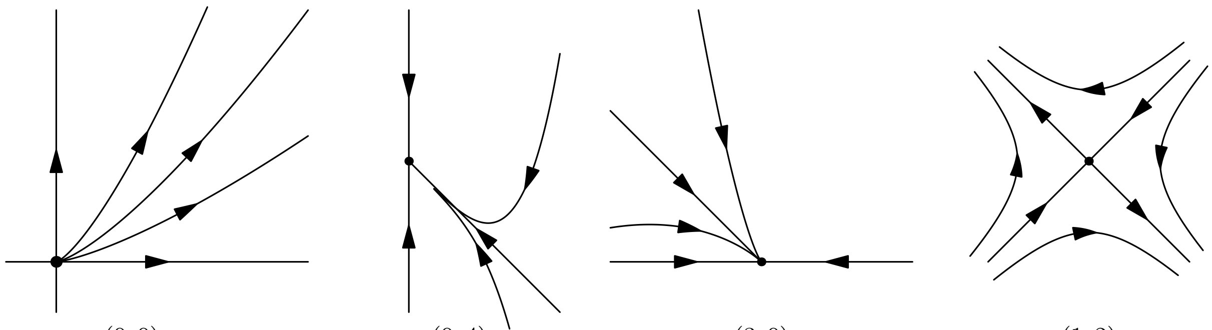



Figure 3.11: Local phase portraits near each of the stationary points.

**Linearisation:**

$$Df(\underline{x}) = \begin{pmatrix} 3-2r-s & -r \\ -2s & 4-2r-2s \end{pmatrix}$$

At $(0,0)$ :

$$Df(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

The eigenvalues are 3 and 4 with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, so it is an unstable node, with almost all solutions tangential to the r-axis and the local solution is as sketched in Figure 3.11).

At $(0,4)$ :

$$Df(0,4) = \begin{pmatrix} -1 & 0 \\ -8 & -4 \end{pmatrix}$$

The eigenvalues are $-1, -4$ so it is a stable node. The eigenvectors are $\underline{e}_{-1} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, so almost all solutions are tangential to $\underline{e}_{-1}$ at the stationary point. See Figure 3.11.

At $(3,0)$ :

$$Df(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -2 \end{pmatrix}$$

so the eigenvalues are $-3$ with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $-2$ with eigenvector $\underline{e}_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. So it is a stable node and almost all solutions are tangential to $\underline{e}_2$ at the stationary point.

At $(1,2)$ :

$$Df(1,2) = \begin{pmatrix} -1 & -1 \\ -4 & -2 \end{pmatrix}$$

so the characteristic equation is $(s+1)(s+2) - 4 = 0$ or $s^2 + 3s - 2 = 0$, i.e. $s_\pm = \frac{-3 \pm \sqrt{17}}{2}$. Since $s_+ > 0$ and $s_- < 0$ is a saddle and the eigenvectors are $\underline{e}_\pm = \begin{pmatrix} -1 \\ s_\pm + 1 \end{pmatrix}$, so $\underline{e}_+$ slopes downwards and $\underline{e}_-$ slopes upwards.

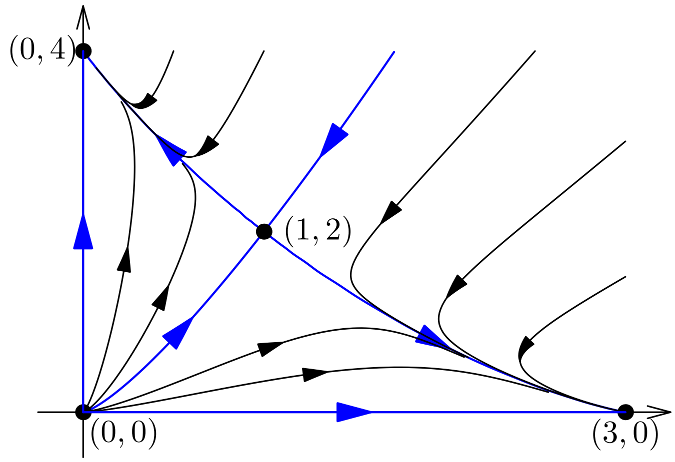



Figure 3.12: Consistent global phase portrait in $r, s \geq 0$. Note the role of the separatrices in separating regions of initial conditions tending to each of the two stable nodes.

Putting the information together suggests the global phase portrait of Figure 3.12. The important features are the separatrices which separates solutions tending to $(0,4)$ from those approaching $(3,0)$.and the tangential approach to approximate stationary points.

**Example 3.5** (ODEs for mutualistic interactions)**.** Imagine a colony of bees ($b$) and flower ($f$) with $b, f \geq 0$ denoting the population size in normalized coordinates so that one unit represents many animals and we are justified in approximating the population size as a continuous variable. Bees fly from flower to flower gathering nectar for food, and the flowers also benefit from the bees for pollination. This is a typical example of mutualistic interaction. A model for their population is

$$\dot{b} = (3 - 3b + f)b, \qquad \dot{f} = (1 + b - f)f. \tag{3.7}$$

All the stationary points are $(0,0), (0,1), (1,0), (2,3)$. By evaluating the Jacobian

$$Df(b,f) = \begin{pmatrix} 3 - 6b + f & b \\ f & 1 + b - 2f \end{pmatrix}$$

we have the following classification:

**(0,0):** unstable node

**(1,0):** saddle node

**(0,1):** saddle node

**(2,3):** stable node

Basically, the stability/instability of any stationary point can be implied from the linearised system, when no eigenvalue has zero real part. These points are called *hyperbolic fixed points*. Otherwise, fixed points with zero real part in their eigenvalues (like centres) are called *non-hyperbolic fixed point*. The behaviours near these stationary points are more difficult to study: while all orbits around centres are periodic, there could be no periodic solutions when nonlinear higher order terms are added.

## 3.4 Stability and Lyapunov functions

We have seen that if $\mathrm{Re}\lambda_i < 0$ for all eigenvalues $\lambda_i$ of $A$, solutions of linear system $\dot{x} = Ax$ converge to the origin. In fact, if $A$ can be diagonised with the eigenpairs $(\lambda_i, e_i)$, then the solution can be written as

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} e_i,$$

while the prefactor $e^{\lambda_i t}$ goes to zero as $t$ goes to infinity. Is this true of the corresponding nonlinear systems near the stationary point, when the solution can not be obtained in explicit form? This is part of a much more general question about stability of stationary points, so first let us introduce some definitions. The main questions to be answered in this subsection are:

(1) How do stability results for linear systems carry over to nonlinear systems locally?
(2) How about the boundedness or stability of solutions for $\dot{x} = f(x), x \in \mathbb{R}^n$.

**Definition 3.1** (Asymptotic stability)**.** A stationary point $x^*$ of an autonomous system $\dot{x} = f(x)$ is asymptotically stable iff there exists an open neighbourhood $U$ of $x^*$ such that $\varphi_t(x_0) \to x^*$ as $t \to \infty$ for all $x_0 \in U$.

If all eigenvalues $\lambda_i$ are negative, then the solution starting at any $x_0$ converges to the origin. If $A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is diagonal, this asymptotical stability can also be shown alternatively by considering the (squared) distance $V(x) = |x|^2$ between the solution $x(t)$ and the origin. Since

$$\frac{d}{dt}V(x) = 2x \cdot \dot{x} = 2\sum_{i=1}^{n} \lambda_i x_i^2 \le 0,$$

the square distance $|x(t)|^2$ is strictly decreasing, until $x(t)$ reaches the origin. This means that $V(x(t)) = |x(t)|^2$ converges to zero .

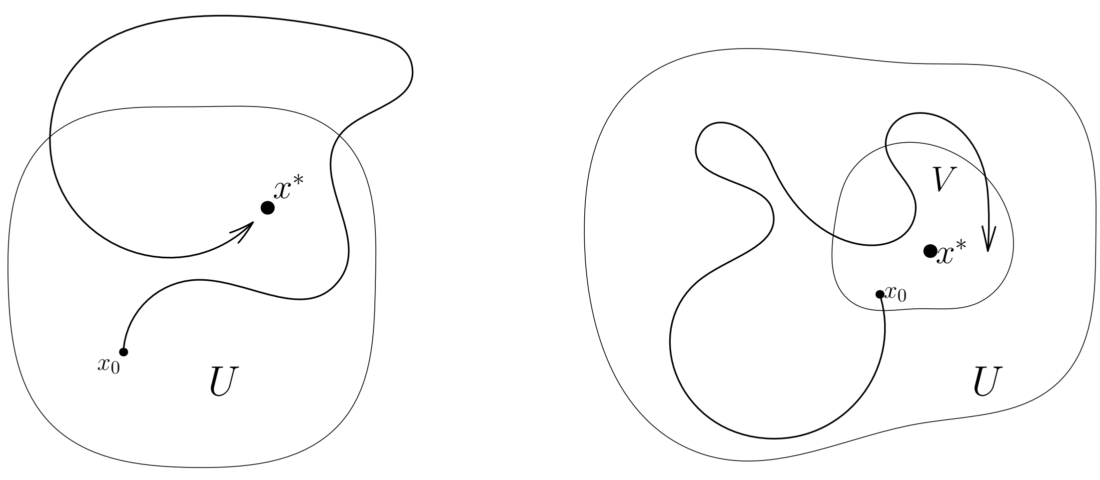



Figure 3.13: (a) Asymptotic stability; (b) Lyapunov stability.

This means that if a solution starts sufficiently close to $x^*$, its solution eventually becomes arbitrary close to $x^*$. But it does not imply the solution stays within $U$ for all $t > 0$.

**Definition 3.2** (Lyapunov stability)**.** A stationary point $x^*$ of an autonomous ODE is Lyapunov stable iff for every open neighbourhood $U$ of $x^*$ there exists an open neighbourhood $W \subset U$ such that $x_0 \in W$ implies $\varphi_t(x_0) \in U$ for all $t \ge 0$.

In other words solutions stay close to $x^*$ if they start close enough to $x^*$. Note that Lyapunov stability does not imply asymptotic stability (think of a linear centre). We have been thinking about linearisation, which motivates our final definition.

**Definition 3.3** (Linear stability)**.** A stationary point $x^*$ of an autonomous ODE is linearly stable iff the real parts of every eigenvalue of $Df(x^*)$ is negative.

For linear systems $\dot{x} = Ax$, if the eigenvalues of $A$ have negative real parts then $|x(t)| \to 0$ as $t \to \infty$, i.e. solutions are asymptotically stable (we have shown this in the special case of distinct eigenvalues). We will take a geometric view of stability (see the textbook by Meiss for a more analytic treatment). The geometric approach is via motivating example for showing solutions are bounded (often an important first step in physics).

The proof of the lemma uses some results from calculus. First recall the chain rule for the derivative of a function of a function. In one dimension

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\mathrm{d}V(x(t))}{\mathrm{d}x}\frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

and in higher dimensions ($x(t) \in \mathbb{R}^n$)

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \nabla V(x(t)) \cdot \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \nabla V(x) \cdot \dot{x}.$$

So if $x$ satisfies the differential equation $\dot{x} = f(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \nabla V(x) \cdot f(x).$$

**Lemma 3.1** (Bounding Lemma)**.** *Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f$ smooth. Suppose there exists a compact set $U \subset \mathbb{R}^n$, $\epsilon > 0$ and a continuously differentiable function $V : U \to \mathbb{R}$ such that*

*(a) $V(x) \geq 0$ in $\mathbb{R}^n \backslash U$*

*(b) $U \subset S_c = \{V(x) \leq c\}$ for all $c \geq c_0 > 0$.*

*(c) $S_c$ is compact and $S_c \subset S_{c'}$ if $c < c'$*

*(d) $\dot{V}(x) < -\varepsilon$ for all $x \in \mathbb{R}^n \backslash U$.*

*then for all $x_0 \in \mathbb{R}^n$ there exists $t_0 > 0$ such that $\varphi_t(x_0) \in S_{c_0}$ for all $t \geq t_0$.*

*Proof.* We first show that $x(t)$ enters $S_{c_0}$ at some time, even initially $x_0$ is not in $S_{c_0}$. If $x_0 \notin S_{c_0}$ then $\dot{V} < -\varepsilon$ and $V(x_0) > c_0$, so

$$V(\varphi_t(x_0)) < V(x_0) - \varepsilon\, t$$

and so there exists $t_0 < \frac{V(x_0) - c_0}{\varepsilon}$ such that

$$V(\varphi_{t_0}(x_0)) = c_0, \qquad \text{i.e. } \varphi_{t_0}(x_0) \in S_{c_0}.$$

Once $x(t)$ is in $S_{c_0}$, we show that it stays in $S_{c_0}$. To make this argument formally, suppose

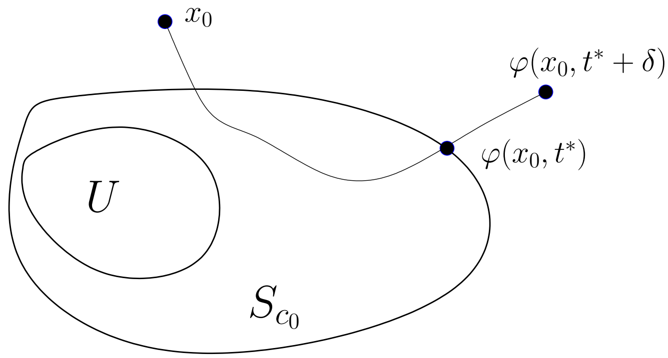



Figure 3.14: What if $\varphi_t(x_0)$ leaves $S_{c_0}$?

$\varphi_{t^*}(x_0) \in \partial S_{c_0}$ and for $\delta > 0$ sufficiently small $\varphi_t(x_0) \notin S_{c_0}$, for any $t \in (t^*, t^* + \delta)$ (see Figure 3.14). Then $V(\varphi_{t^*+\delta}(x_0)) > c_0 = V(\varphi_{t^*}(x_0))$ and $\dot{V}(\varphi_t(x_0)) < -\epsilon$ for any $t \in (t^*, t^* + \delta)$. Then

$$0 < V(\varphi_{t^*+\delta}(x_0)) - V(\varphi_{t^*}(x_0)) = \int_{t^*}^{t^*+\delta} \frac{\mathrm{d}}{\mathrm{d}t} V(\varphi_t(x_0))\mathrm{d}t < \int_{t^*}^{t^*+\delta} (-\epsilon)\mathrm{d}t = -\delta\epsilon.$$

So we get a contradiction. □

**Example 3.6** ( Lorenz Equations)**.** The Lorenz equations are three coupled ODEs introduced as a simple model of the weather. They are one of the earliest examples which appeared to behave chaotically. The equations are

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy$$

with $\sigma, r, b \geq 0$. We wish to show that all solutions are bounded.

Need to 'create' a positive function $V(x, y, z)$ with the properties of the lemma. The obvious thing to do is to choose quadratic function, slightly less obvious is which quadratic function. The following approach is not optimal but works!

Set

$$V(x, y, z) = Ax^2 + By^2 + C(z - 2r)^2$$

and we want to choose the positive constants A,B,C to satisfy the lemma. So

$$\begin{aligned}\frac{1}{2}\dot{V} &= Ax\dot{x} + By\dot{y} + C\dot{z}(z - 2r)\\ &= A\sigma x(y - x) + By(rx - y - xz) + C(-bz + xy)(z - 2r)\end{aligned}$$

where we have used the ODE to replace $\dot{x}, \dot{y}$, and $\dot{z}$. We want $\dot{V} < 0$ for large enough $x, y$ and $z$, so the 'difficult' terms are cross multiplications like $xy$ and $xyz$ (good terms are quadratic terms, with negative coefficients). Choose the constants to remove them i.e.

$$\begin{aligned} B - C &= 0 \qquad && (xyz \text{ terms vanish }) \\ A\sigma + Br - 2Cr &= 0 \qquad && (xy \text{ terms vanish}) \end{aligned}$$

So set $B = C$ and $A\sigma = Br$, e.g. $A = r, B = C = \sigma$ (we have one degree of freedom to choose these constants: $A = r/\sigma, B = C = 1$ works as well), then

$$V(x, y, z) = \sigma r x^2 + \sigma y^2 + \sigma(z - 2r)^2$$

and

$$\frac{1}{2}\dot{V}(x,y,z) = -\sigma r x^2 - \sigma y^2 - \sigma(bz^2 - 2rbz) = -\sigma r x^2 - \sigma y^2 - \sigma b(z-r)^2 + \sigma b r^2.$$

Notice it is $(z-r)^2$ on the right hand side, instead of $(z-2r)^2$ as in $V$.

The specific form of $\dot{V}/2$ also tells us which set to look at. So provided that $(x,y,z)$ lies outside the set

$$\tilde{U} = \left\{(x,y,z) | \sigma r x^2 + y^2 + \sigma b(z-r)^2 \le br^2\right\} \qquad \text{(an ellipsoid)}$$

then $\dot{V} < 0$. It is possible to modify the Bounding Lemma to deal with this, but the actual conditions of the bounding lemma require $\dot{V} < -\epsilon < 0$. Thus we choose $U$ a little larger: pick $\epsilon > 0$ and set

$$U = \{(x,y,z) | \sigma r x^2 + \sigma y^2 + \sigma b(z-r)^2 \le \sigma b r^2 + \epsilon\}$$

and so in $\mathbb{R}^3 \backslash U$

$$\dot{V} = -r\sigma x^2 - \sigma y^2 - \sigma b(z-r) + \sigma b r^2 \le -2\epsilon < -\epsilon.$$

Now just choose $c_0$ sufficiently large that

$$S_{c_0} = \{(x,y,z) | r x^2 + \sigma y^2 + \sigma b(z-2r)^2 \le c_0\}$$

(another ellipsoid) contains $U$.

*Remark.* We can relate $\dot{V}$ to $V$ in a differential inequality, to induce some information about the behaviour of $V$. More precisely, the above expression for $\dot{V}$ implies that

$$\dot{V}(x,y,z) = -2r\sigma x^2 - 2\sigma y^2 - \sigma b(z-2r)^2 - \sigma b z^2 + 4\sigma b r^2 \le -\mu V(x,y,z) + 4\sigma b r^2.$$

with $\mu = \min(2\sigma, 2, b)$. The advantage here is that we can "solve" this differential inequality. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\mu t} V(x,y,z)\right) = e^{\mu t}\left(\dot{V} + \mu V\right) \le 4\sigma b r^2 e^{\mu t}.$$

Integrating both sides for time from zero to $t$, then

$$e^{\mu t} V(x,y,z) - V(x_0,y_0,z_0) \le 4\sigma b r^2 \int_0^t e^{\mu\tau} \mathrm{d}\tau = \frac{4\sigma b r^2}{\mu}(e^{\mu t} - 1).$$

That is

$$V(x,y,t) \le e^{-\mu t} V(x_0,y_0,z_0) + \frac{4\sigma b r^2}{\mu}\left(1 - e^{-\mu t}\right) \le e^{-\mu t} V(x_0,y_0,z_0) + \frac{4\sigma b r^2}{\mu},$$

which is bounded for any $t \ge 0$. We can estimate the time when the trajectories enter the set (different from the above one)

$$\bar{U} = \left\{(x,y,z) | V(x,y,z) \le \frac{4\sigma b r^2}{\mu} + \epsilon\right\},$$

that is $t \ge \frac{1}{\mu}\log\frac{\epsilon}{V(x_0,y_0,z_0)}$.

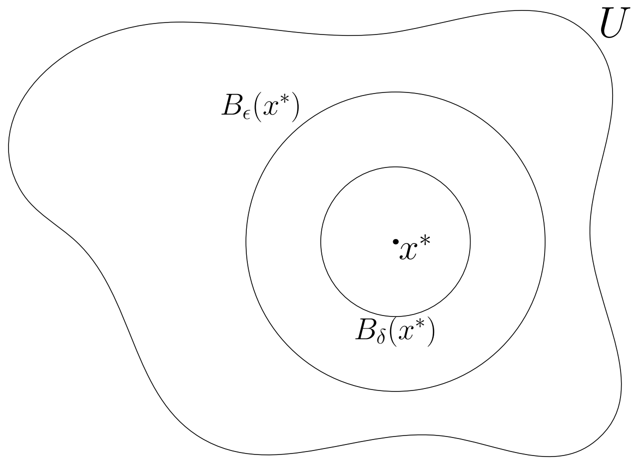



Figure 3.15: Regions defined in proof.

*Remark.* The function $V$ above is not unique, and we can choose alternative ones (along with many others) like

$$V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2.$$

Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -2\sigma x^2 - 2y^2 - b(z - r - \sigma)^2 - bz^2 + b(r + \sigma)^2 \leq -\alpha V + b(r + \sigma)^2,$$

where $\alpha = \min(2\sigma, 2, b)$. Then we can choose the set to be

$$\tilde{U} = \left\{ (x, y, z) \mid x^2 + y^2 + (z - \sigma - r)^2 \leq \frac{b(r + \sigma)}{\alpha} \right\},$$

or relax the right hand side by changing it to $b(r + \sigma)^2/\alpha + \epsilon$.

The same basic idea works for stationary points.

**Definition 3.4** (Lyapunov functions)**.** A function $V : U \to \mathbb{R}$ is called a Lyapunov function on $U \subseteq \mathbb{R}^n$ iff it is continuously differentiable, $V(x) \geq 0$ on $U$ and $\dot{V} \leq 0$ on $U$.

**Theorem 3.2** (Lyapunov's Stability Theorem)**.** *Suppose $x^* \in \mathbb{R}^n$ is a stationary point of $\dot{x} = f(x)$ with $f$ smooth. Let $U$ be an open neighbourhood of $x^*$ and suppose there exists a Lyapunov function $V : U \to \mathbb{R}$ such that $V(x) > 0$ on $U \setminus \{x^*\}$ and $V(x^*) = 0$. Then $x^*$ is a Lyapunov stable. If in addition $\dot{V} < 0$ in $U \setminus \{x^*\}$ then $x^*$ is asymptotically stable.*

*Proof.* Choose $\varepsilon > 0$ small enough so that $\{x \mid |x - x^*| \leq \varepsilon\}$ lies entirely in $U$, and let $c_0 = \min_{|x-x^*|=\varepsilon} V(x)$ which exists as $|x - x^*| = \varepsilon$ is compact (closed and bounded), and $c_0 > 0$ as $x^* \notin \{x \mid |x - x^*| = \varepsilon\}$. Let $\mathcal{B}_\varepsilon(x^*) = \{x \mid |x - x^*| < \varepsilon\}$. Now $V$ is continuous, and $V(x^*) = 0$, so there exists $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(x^*), V(x) < \frac{1}{2}c_0$.

Consider $x_0 \in \mathcal{B}_\delta(x^*)$. Since $\dot{V} \leq 0$ in $U$, $V(\varphi_t(x_0)) \leq V(x_0) < \frac{1}{2}c_0$ for all $t$ such that $\varphi_t(x_0) \in U$, and hence $V(\varphi_t(x_0)) < \frac{1}{2}c_0 < c_0$, the minimum on $|x - x^*| = \varepsilon$. Hence $\varphi_t(x_0) \in \mathcal{B}_\varepsilon(x^*)$ for all $t > 0$.

Suppose in addition that $\dot{V} < 0$ if $x \in U\setminus\{x^*\}$, (note that $\dot{V}(x^*) = 0$ since $x^*$ is stationary). Then if $x_0 \in \mathcal{B}_\delta(x^*)$ as before, $V(\varphi_t(x_0))$ is strictly decreasing and hence tends to a limit $\bar{V} = \lim_{t\to\infty} V(\varphi_t(x_0))$. At the limit $\dot{V} = 0$ hence the limit must be $x^*$. □

**Example 3.7.** Consider the Lorenz equations with $0 < r < 1$.

$$\dot{x} = \sigma(y - x); \qquad \dot{y} = rx - y - xz; \qquad \dot{z} = -bz + xy$$

with $\sigma, b \geq 0$ and $0 < r < 1$. Try a Lyapunov function of the form

$$V(x, y, z) = Ax^2 + By^2 + Cz^2,$$

with $\frac{1}{2}\dot{V} = A\sigma x(y - x) + By(rx - y - xz) + Cz(-bz + xy)$. Choose $B = C$ to remove the $xyz$ terms:

$$\frac{1}{2}\dot{V} = A\sigma(-x^2 + xy) + B(rxy - y^2 - bz^2)$$

We deal with the $xy$ terms to ensure $\dot{V}$ is negative, by choosing $A$ and $B$ appropriately (matching the square!). Set $B = \sigma$, $A = r$ so all terms together

$$\begin{aligned}\frac{1}{2}\dot{V} &= r\sigma(-x^2 + 2xy) - \sigma y^2 - \sigma b z^2 \\ &= r\sigma(-(x - y)^2) - (\sigma - r\sigma)y^2 - \sigma b z^2 \\ &= -r\sigma(x - y)^2 - \sigma(1 - r)y^2 - \sigma b z^2 \\ &< 0 \qquad\qquad \text{if } (x, y, z) \neq (0, 0, 0)\end{aligned}$$

if $0 < r < 1$. Hence the origin is asymptotically stable when $0 < r < 1$.

**Example 3.8.** Let us consider a degenerate node with repeated eigenvalues $\lambda = -2$:

$$\dot{x} = -2x + y, \quad \dot{y} = -2y.$$

Set $V(x, y) = x^2 + By^2$ and $B$ is chosen later. Then

$$\frac{1}{2}\dot{V} = x(-2x + y) + By(-2y) = -2x^2 + xy - 2By^2 = -2(x - \frac{1}{4}y)^2 + \frac{1}{8}y^2 - 2By^2.$$

So for any $B > \frac{1}{16}$ the function is a Lyapunov function and the origin is asymptotically stable.

In many practical examples, the stationary point $x^*$ can still be asymptotic stable, when the condition $\dot{V} < 0$ in $U \subset \{x^*\}$ in **Theorem 3.2** can be relaxed to $\dot{V} \leq 0$, provided that $x^*$ is the only fixed point.

**Theorem 3.3** (LaSalle's Invariance Principle)**.** *Suppose that $V : U \subset \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $\dot{x} = f(x)$. If the set $\{x \in U \mid \dot{V}(x) = 0\}$ contains only one fixed point $x^*$, then $x^*$ is asymptotically stable.*

**Example 3.9.** Consider the equation $\ddot{x} + \mu\dot{x} + \omega^2 x = 0$ descibing the motion of a harmonic oscillator with friction ($\mu > 0$), or equivalently $\dot{x} = y, \quad \dot{y} = -\mu y - \omega^2 x$. If we choose $V(x, y) = \omega^2 x^2 + y^2$, then $\dot{V} = -2\mu y^2 \leq 0$. Since $(0, 0)$ is the only fixed point in the set $\{(x, y) \mod \dot{V}(x, y) = 0\} = \{(x, y) \mid y = 0\}$, by Lasalle's invariance principle, the origin is asymptotically stable. In other words, the harmonic oscillator will eventially stop moving.

## 3.5 Linearisation and nonlinear terms

The previous result shows that if $\mathrm{Re}(\lambda) < 0$ and the eigenvalues are distinct, then a linearly stable fixed point is locally stable when nonlinear terms are added back in. This is an example of a range of persistence results for behaviour.

**Example 3.10.** The linearised system of the full nonlinear system

$$\dot{x} = -x, \quad \dot{y} = y + x^2$$

is $\dot{x} = -x, \dot{y} = y$, with two eigenvalues $\pm 1$. The phase portrait for this saddle system should be well known now, but there are two special straight lines deserving more attention: the $x$-axis and and the $y$-axis. If the initial condition $(x_0, y_0)$ is on the $x$-axis (that is $y_0 = 0$), then the solution $(x(t), y(t))$ converges to the origin, as time $t$ goes to infinity. This linear space is called the *stable manifold*, denoted as $E^s$. Although the solution with initial condition away from the $x$-axis goes not infinity, as time $t$ goes to positive infinity, initial condition $(x_0, y_0)$ resides on the $y$-axis has the special property that its solution converges to the origin as time $t$ goes to *negative infinity*. The $y$-axis is called *unstable manifold* (which is the stable manifold when time is reversed), denoted as $E^u$.

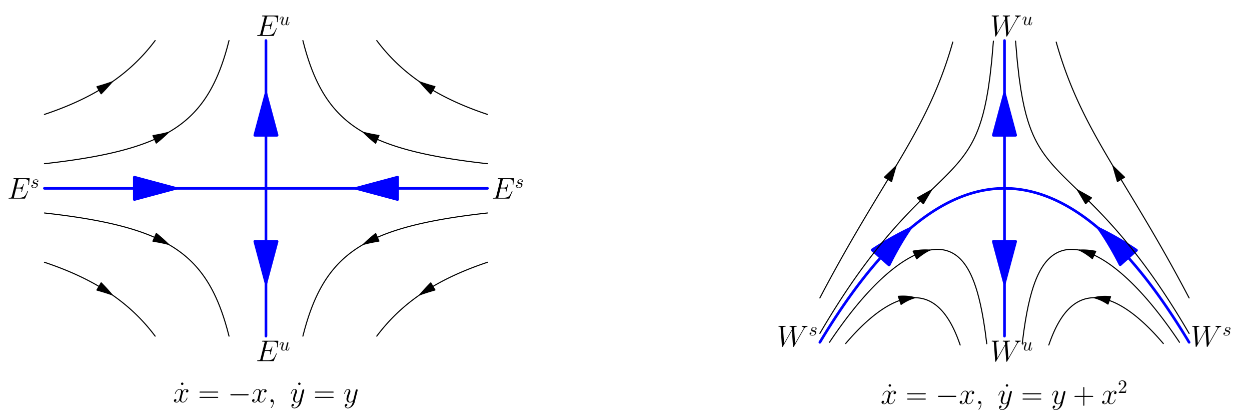



Figure 3.16: Invariant manifolds $W^s$ and $W^u$ for the linearised system and full nonlinear system in Example 3.10.

Back to the nonlinear system, the correponding phase portrait is *deformed* from that of the linearised system. There are also special curves, the stable manifold $W^s$ and the unstable manifodl $W^u$, whose solution converges to the origin, as time $t$ goes to positive infinity and negative infinity, respectively. In general, $W^s$ (or $W^u$) is no longer straight line, but it is tangent to $E^s$ (or $E^u$) at the origin.

In fact, in this example, we can find the equations for $W^s$ and $W^u$. Obviously, $W^u$ is the $y$-axis: if $x_0 = 0$, then the ODE $\dot{x} = -x$ implies $x(t) = x_0 e^{-t} \equiv 0$, and $y(t) = y_0 e^t \to 0$ as $t \to -\infty$. The stable manifold is $W^s = \{(x, y) \mid y = -x^2/3\}$. If $(x_0, y_0)$ is on $W^s$, that is $y_0 = -x_0^2/3$, then $x(t) = x_0 e^{-t}$ and $y$ satisfies the linear ODE

$$\dot{y} = y + x_0^2 e^{-2t}.$$

From the fact that $\dfrac{\mathrm{d}}{\mathrm{d}t}(e^{-t}y) = e^{-t}(\dot{y} - y) = x_0^2 e^{-3t}$, we get

$$e^{-t}y(t) = y_0 + \int_0^t x_0^2 e^{-3\tau} \mathrm{d}\tau = y_0 + \frac{x_0^2}{3}(1 - e^{-3t}) = \left(y_0 + \frac{x_0^2}{3}\right) - \frac{x_0^2}{3}e^{-3t} = -\frac{x_0^2}{3}e^{-3t}.$$

Since the solution $(x, y) = \left(x_0 e^{-t}, -\frac{x_0^2}{3} e^{-2t}\right)$ stays on $W^s$ and converges to the origin (as $t$ goes to positive infinity), $W^s$ is a stable manifold.

*Remark.* Here the word *manifold* is used for a smooth geometric curve or surface: if the manifold is one dimension, it is a smooth curve (inluding straight lines); if the manifold is two dimension, it is a smooth surface (including planes). But normally we do not know the dimension of the curve or the surface in advance, so it is better to use the generic name *manifold* instead of the more common curve or surface.

This above relationship between full system and its linearised system is summarised in the following theorem, provided that none of the eigenvalue has zero real part.

**Theorem 3.4** (Stable Manifold Theorem)**.** *Suppose $\dot{x} = Ax + O(x^2)$ and $A$ has no eigenvalues with $\mathrm{Re}(\lambda) = 0$ ($x = 0$ is called a* hyperbolic *stationary point in this case). Then after a change of coordinates the system is*

$$\dot{x}_1 = A^+ x_1 + O(|x|^2), \qquad \dot{x}_2 = A^- x_2 + O(|x|^2)$$

*where $A^+$ has $\mathrm{Re}(\lambda) > 0, A^-$ has $\mathrm{Re}(\lambda) < 0$. Moreover there are invariant manifolds $W^u$ and $W^s$ with*

$$W^u = \{x \in U \mid \varphi_t(x) \to 0 \text{ as } t \to -\infty\}$$

*and*

$$W^s = \{x \in U \mid \varphi_t(x) \to 0 \text{ as } t \to \infty\}.$$

*which are of of the same dimension as $x_1$ (resp. $x_2$) and tangential to $x_2 = 0$ (resp. $x_1 = 0$) at the origin.*

This implies the *persistence of saddle structure* near a stationary point when the nonlinear terms are added back into the linearisation. The correponding changes in the phase portraits or the deformation of the stable/unstable manifolds are best discribed using language from *topology* (think about the deformation of a coffee mug into a donut). The persistence between structure are called *topologically conjugate* or *topologically equivalent*, but we will omit this complicated topological language and keep a mental picture instead, as in the following theorem.

**Theorem 3.5** (Hartman–Grobmann)**.** *If $\dot{x} = Ax + O(x^2)$ and $A$ has no eigenvalue with zero real part, then the behaviour near the neighbourhood of the origin is topologically equivalent to the linear system $\dot{x} = Ax$.*

**Example 3.11.** Now consider the system $\dot{x} = y, \dot{y} = x^2 + x$ and its linearised system $\dot{x} = y, \dot{y} = x$. The trajectories, governed by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + x}{y}$(which is separable), are given by $\dfrac{x^3}{3} + \dfrac{x^2 - y^2}{2} = C$. The stable manifold $W^s$ concides with the unstable manifold $W^u$, and is $\dfrac{x^3}{3} + \dfrac{x^2 - y^2}{2} = 0$.

If $A$ has eigenvalue with zero real part, then the situation is much more complicated, as we can see from the following two examples.

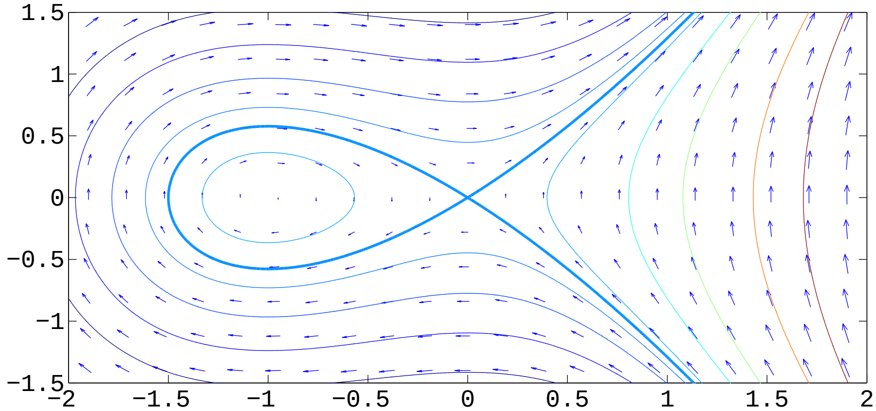



Figure 3.17: The phase portrait for $\dot{x} = y, \dot{y} = x^2 + x$ in Example 3.11.

**Example 3.12.** Consider the system $\dot{x} = x, \dot{y} = y^2$, which you can solve the individual equations separately (they are de-coupled from each other). The lienarized system (near the origin) $\dot{x} = x, \dot{y} = 0$ has only horizontal phase curves. For a related system $\dot{x} = y^2, \dot{y} = x$ (you can also solve this explicitly).

If there is zero eigenvalue in the linearised system, their local behaviours of the full nonlinear system near the fixed point are different, as we can see from Figure 3.18 and 3.19.

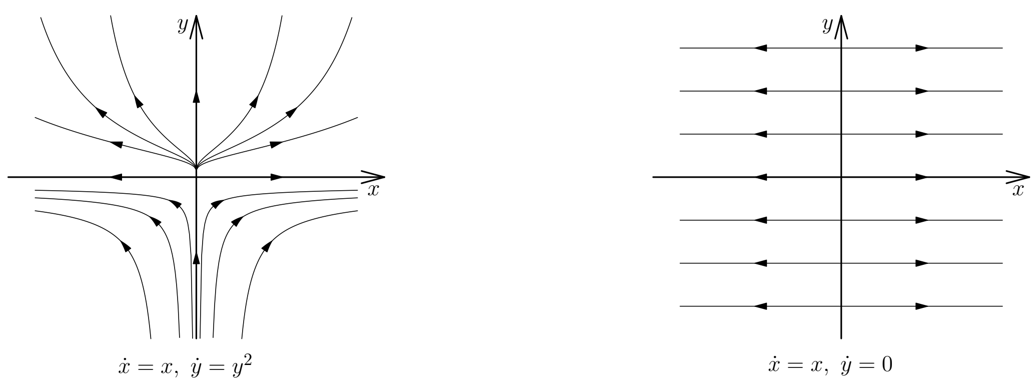



Figure 3.18: Phase portraits for the system $\dot{x} = x, \dot{y} = y^2$ with zero eigenvalues at the origin.

*Exercise.* Find the solution to the system

$$\dot{x} = x, \qquad \dot{y} = y^2$$

and

$$\dot{x} = y^2, \quad \dot{y} = x$$

whose phase portraits are given in Figure 3.18 and 3.19.

The situation is even worse if $f(x) = O(|x|^2)$, because the linearised system $\dot{x} = 0$ does not tell anything about the behaviour of the system near the origin. You can see two examples in Figure 3.20. Information can still be obtained in certain cases, by looking at the trajectories, or by transforming into polar coordinates.

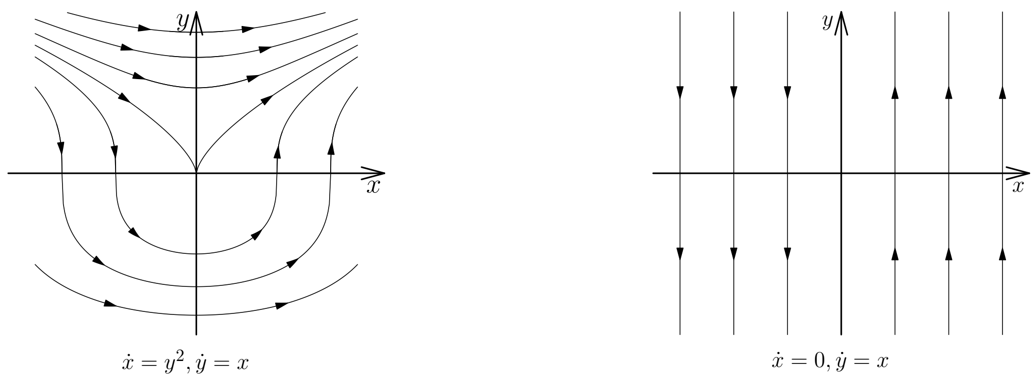



Figure 3.19: Phase portraits for the system $\dot{x} = y^2, \dot{y} = x$ with zero eigenvalues at the origin.

**Example 3.13.** Consider the system $\dot{x} = -xy, \dot{y} = x^2 + y^2$, whose phase portrait is given in Figure 3.20(left figure). The trajectory, governed by the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2+y^2}{xy}$ is homogeneous. By the change of variable $y = zx$, the ODE becomes $\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{2z^2+1}{z}x$, which is separable. The solution is given by $x^4(2z^2 + 1) = C$ and the trajectories are given by $2x^2y^2 + x^4 = C$.

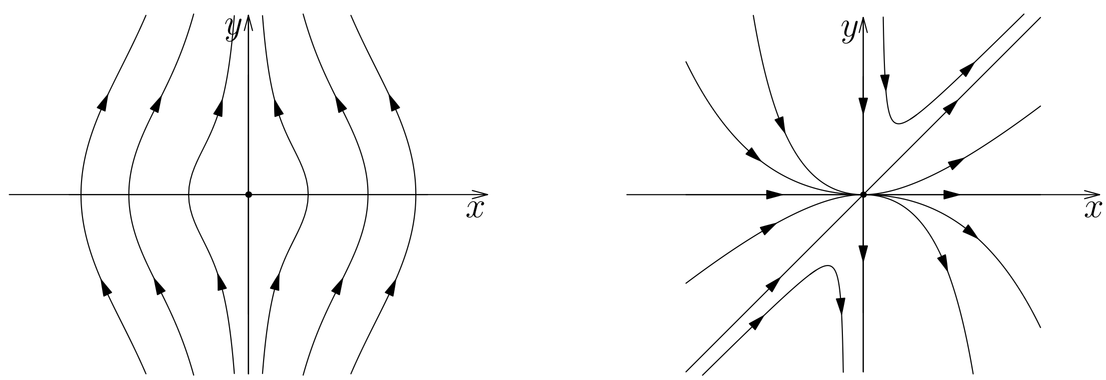



Figure 3.20: Phase portraits for systems with zero linear parts at the origin: Left: $\dot{x} = -xy, \dot{y} = x^2 + y^2$; Right: $\dot{x} = x^2, \dot{y} = y(2x - y)$.

## 3.6 Maps

Besides continuous dynamical systems using differential equations, discrete dynamical systems defined by maps are also popular in modelling (Fibonacci number for the population of animals), taking the form $x_{n+1} = f(x_n)$ with $x \in \mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}^k$. These systems are easier to deal with numerically, but more difficult analytically (precisely because of the discrete phase space). If there are parameters in the map, we write

$$x_{n+1} = f(x_n, \mu),$$

with $\mu \in \mathbb{R}^m$, $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$. Given an initial condition $x_0$, the trajectory is the sequence

$$(x_0, x_1, x_2, \dots),$$

i.e, a discrete set of points in phase space $\mathbb{R}^n$.

We often write

$$x_n = f(x_{n-1}) = f(f(x_{n-2})) = \underbrace{f(f(\cdots(f(}_{n \text{ times}} x_0))\cdots)) = \underbrace{f \circ f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0),$$

where $f^n$ means the $n^{th}$ iteration of $f$, which is **NOT** $[f(x_0)]^n$, the $n$-th power of $f(x_0)$.

Similar to their continuous counterparts, we are mainly interested in the *fixed points* and *periodic orbits* of maps to understand their local behaviours, as a starting point for more complicated situations. A ***fixed point*** of the discrete dynamical system $x_{n+1} = f(x_n)$ is a solution of

$$x = f(x).$$

***Periodic orbits*** are defined similarly: they satisfy

$$x_{n+p} = x_n \quad \text{for all } n \geq 0,$$

and $p$ is called the period. Any point in the periodic orbit with period $p$ is just a fixed point of $f^p$, that is,

$$x = f^p(x) = \underbrace{f \circ f \circ \cdots f}_{p \text{ times}}(x)$$

another algebraic equation! A period-$p$ orbit is usually listed as a sequence of $p$ points $(x_1^*, x_2^*, \cdots, x_p^*)$ such that

$$x_2^* = f(x_1^*), \quad x_3^* = f(x_2^*), \quad \cdots, \quad x_1^* = f(x_p^*),$$

while each of the point $x_k^*$ satisfies $x = f^p(x)$. Because of the periodicity, the period-$p$ orbit $(x_2^*, x_3^*, \cdots, x_p^*, x_1^*)$ is the same as $(x_1^*, x_2^*, \cdots, x_p^*)$, and we only need to choose one orbit out of the $p$ equivalent ones.

As with ODEs, we are interested in qualitative properies like special solutions and their stabilities, invariant sets, long term behaviours and the dependence of these properties on parameters.

**Example 3.14.** Consider the simplest linear map $x_{n+1} = ax_n + b$. If $a = 1$, then $x_n = x_{n-1} + b = \cdots = x_0 + nb$ and there is no fixed point, unless $b = 0$. Otherwise if $a \neq 1$, the only fixed point is $x^* = b/(1-a)$. From the fact that

$$x_n - x^* = ax_{n-1} + b - \frac{b}{1-a} = ax_{n-1} - \frac{ab}{1-a} = a\left(x_{n-1} - x^*\right),$$

we get $x_n - x^* = a^n(x_0 - x^*)$ and

$$x_n = x^* + a^n(x_0 - x^*) = a^n x_0 + \frac{1-a^n}{1-a} b.$$

It is also easy to check that, if $a \neq 1$, there is no non-trivial period-2 orbits (check it!)—any period-2 orbit $(x_1, x_2)$ satisfies $x_1 = x_2 = x^*$.

**Example 3.15** (Compound interest)**.** Let $P_n$ be the principal at $n$-th month with initial principal $P_0$, monthly interest rate $r$ and monthly payment $M$, then $P_n$ satisfies the relation

$$P_{n+1} = (1+r)P_n - M.$$

From the previous example, we get (with $a = 1 + r, b = -M$)

$$P_n = (1+r)^n P_0 - \frac{M}{r}\big((1+r)^n - 1\big).$$

**Example 3.16** (Circle map)**.** In unimodal map, we are only interested in the fractional part of the numbers, which can be identified by taking the integer part out by `mod 1` operation. For example, $1.1 = -0.9 = 0.1 (\text{mod } 1)$. The simplest unimodal maps is

$$x_{n+1} = mx_n + b \text{ (mod 1)},$$

where $m$ is usually an integer (so that you get the same $x_{n+1}$ if $x_n$ is replaced by $x_n + \ell$ for any integer $\ell$). Consider the map $x_{n+1} = 3x_n(\text{mod}1)$, then any fixed point $x^*$ satisfies $x^* = 3x^* + \ell$ for some integer $\ell$, or $x^* = \ell/2$. Therefore, $x^* = 0$ or $x^* = 1/2$ (any other $\ell$ leads to either fixed point).

Any period-2 orbits $(x_1^*, x_2^*)$ satisfies

$$x_2^* = 3x_1^* + \ell_1, \quad x_1^* = 3x_2^* + \ell_2$$

or $x_1^* = 9x_1^* + \ell$ (with $\ell = 3\ell_1 + \ell_2$, another integer). Therefore, $x_1^* = \ell/8, \ell = 0, 1, \cdots, 7$. If $x_1^* = 0$ or $x^* = 4/8 = 1/2$, then $x_2^* = 3x_1^* = x_1^*$ and the corresponding period-2 orbits are actually fixed point. Otherwise, we get three non-trivial period-two orbits

$$\left(\frac{1}{8}, \frac{3}{8}\right), \qquad \left(\frac{1}{4}, \frac{3}{4}\right), \qquad \left(\frac{5}{8}, \frac{7}{8}\right),$$

where $\left(\frac{3}{8}, \frac{1}{8}\right)$ is taken as the same periodic orbit as $\left(\frac{1}{8}, \frac{3}{8}\right)$ (similarly for other two).

*Remark.* For a given continuous dynamical system (i.e., the ODE $\dot{x} = f(x)$), we can define an discrete dynamical system in the following ways shown in Figure 3.21. For any given time interval $T > 0$, we can take $x_n = x(nT)$, the solution of the ODE at $t = nT$. Then the sequence $(x_0, x_1, x_2, \cdots)$ is a dynamical system. Alternatively, we can define the discrete points at the intersection of $x(t) \in \mathbb{R}^n$ with a $n-1$ dimensional surface, called *return maps* or *Poincaré maps*.

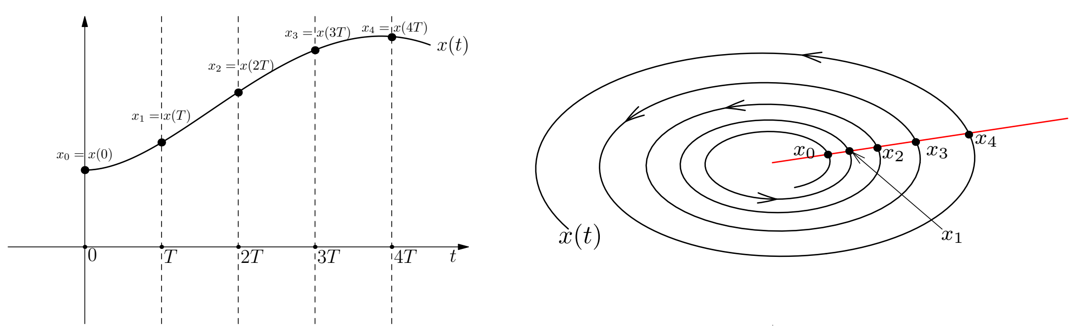



Figure 3.21: Two ways to get discrete dynamical systems from continuous ones, either by $x_n = x(nT)$ or the return map.

Maps also appear in the numerical approximations of ODEs. For example, if we want to consider the solution of $\dot{x} = x(1-x)$ at time $t = 0, h, 2h, \cdots$ ($h$ is called the *time step*, which is usually small) and denote $x_n \approx x(nh)$, then by Taylor expansion,

$$x_{n+1} = x(nh + h) = x(nh) + hx'(nh) + \frac{h^2}{2!}x''(nh) + \cdots = x_n + h(1 - x_n)x_n + O(h^2).$$

Therefore, to the leading order, we get the discrete map $x_{n+1} = x_n + hx_n(1 - x_n)$.

Similarly, for discrete dynamical system governed by $x_{n+1} = f(x_n)$, a set $\Lambda \subseteq \mathbb{R}^n$ is *invariant* iff $x_0 \in \Lambda$ implies $x_n \in \Lambda$ for all $n \geq 0$. In fact, to show $\Lambda$ is invariant, we only need to show that, if $x_n \in \Lambda$, then $x_{n+1} \in \Lambda$.

**Example 3.17** (Logistic map)**.** The logistic map $x_{n+1} = \mu x_n(1-x_n)$ is the simplest discrete dynamical system exhibiting chaotic behaviours (for some parameters of $\mu$). We will study in detail how these behaviours and the associated bifurcations depend on the parameter $\mu$. We can show that the interval $\Lambda = [0,1]$ is invariant when $\mu \in [0,4]$. In fact, for $x_n \in [0,1]$ and $\mu \in [0,4]$, then $x_{n+1} = \mu x_n(1-x_n) \geq 0$ and

$$x_{n+1} = \mu(x_n - x_n^2) = \mu\left[\frac{1}{4} - \left(x_n - \frac{1}{2}\right)^2\right] \leq \frac{\mu}{4} \leq 1.$$

**Example 3.18** (2D system)**.** Consider the system

$$x_{n+1} = x_n\, f(y_n), \qquad y_{n+1} = g(x_n, y_n).$$

The line $x = 0$ is invariant ($x_n = 0 \implies x_{n+1} = 0$), and on $x = 0$, $y_{n+1} = g(0, y_n)$.

**Linear Maps:** The local behaviours of discrete maps can also be inferred from linearisation near fixed points. Let $x^*$ be a fixed point of $x_{n+1} = f(x_n)$ with $x^* = f(x^*)$. If $x$ is close to $x^*$ such that $y_n = x_n - x^*$ is small, then

$$y_{n+1} = x_{n+1} - x^* = f(x_n) - f(x^*) \approx Df(x^*)(x - x^*) = Df(x^*)y_n.$$

Therefore, the linearised equation is

$$y_{n+1} = Ay_n,$$

with the constant matrix $A = Df(x^*)$. The solution can be written as

$$y_n = A^n y_0.$$

If the matrix can be diagonalised as $A = S\Lambda S^{-1}$ (the columns of $S$ are eigenvectors of $A$), then $A^n = S\Lambda^n S^{-1}$. The change of variable $z_n = S^{-1}y_n$ leads to the *normal form*

$$z_{n+1} = S^{-1}y_{n+1} = S^{-1}Ay_n = S^{-1}AS(S^{-1}z_n) = \Lambda z_n,$$

where the matrix power $\Lambda^n$ in the solution $z_n = \Lambda^n z_0$ can be calculated easily. For example,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \mapsto \Lambda^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_m^n \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mapsto \Lambda^n = \begin{pmatrix} \lambda^n & (n+1)\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

The general solution of $x_{n+1} = Ax_n$ when $A$ has $m$ distinct eigenvalues $\lambda_m$ with eigenvectors $e_m$ is

$$x_n = \sum_{j=1}^{m} c_j \lambda_j^n e_j.$$

Here the coefficients $c_j$ are determined from the initial condition ($n = 0$ in the previous equation)

$$x_0 = \sum_{j=1}^{m} c_j e_j.$$

We can start with the simplest case to motivate the criteria of stability. For linear ODES, the canonical example is the scalar ODE $\dot{x} = \lambda x$ with solution $x(t) = x_0 e^{\lambda t}$. Therefore, the stability of the ODE is determined by $\exp(\lambda t)$ as $t$ goes to infinity, or equivalently the boundary $\mathrm{Re}\lambda = 0$. Similarly, if we look at the simplest map $x_{n+1} = \lambda x_n$, then $x_n = \lambda^n x_0$. The stability is determined by $\lambda^n$ as $n$ goes to infinity, or equivalently the boundary $|\lambda| = 1$ ($|\lambda| < 1$ implies stability in the corresponding eigenspace). Now we can proceed for general cases in general dimensions.

**Example 3.19** (Saddle)**.** In Normal Form coordinates, the map is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad \text{with} \qquad |\lambda_1| < 1 < |\lambda_2|.$$

These two components $x_{n+1} = \lambda_1 x_n, y_{n+1} = \lambda_2 y_n$ can be solved explicitly, to give

$$x_n = \lambda_1^n x_0, \qquad\qquad y_n = \lambda_2^n y_0.$$

We can take modulus on the solutions,

$$\frac{|x_n|}{|x_0|} = |\lambda_1|^n, \; \frac{|x_n|}{|x_0|} = |\lambda_1|^n.$$

That is, the solution $(x_n, y_n)$ lies on the generalised hyperbola

$$\left\{ (x, y) \,\middle|\, \left| \frac{x}{x_0} \right| = \left| \frac{y}{y_0} \right|^{\ln|\lambda_1|/\ln|\lambda_2|} \right\}.$$

The motion of these hyperbolas is discrete; an orbit hops along the relevant curve or curves as shown in Figure 3.22. If an eigenvalue is negative then the orbit of a point will oscillate between negative and positive values in that eigen-direction as indicated in Figure 3.22(b).

**Example 3.20** (Focus)**.** The map in normal form is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

The geometric interpretation is clearer if we write the coefficient matrix as

$$\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} = \sqrt{\rho^2 + \omega^2} \begin{pmatrix} \frac{\rho}{\sqrt{\rho^2+\omega^2}} & -\frac{\omega}{\sqrt{\rho^2+\omega^2}} \\ \frac{\omega}{\sqrt{\rho^2+\omega^2}} & \frac{\rho}{\sqrt{\rho^2+\omega^2}} \end{pmatrix} = \underbrace{\lambda}_{\text{dilation}} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation by } \theta}$$

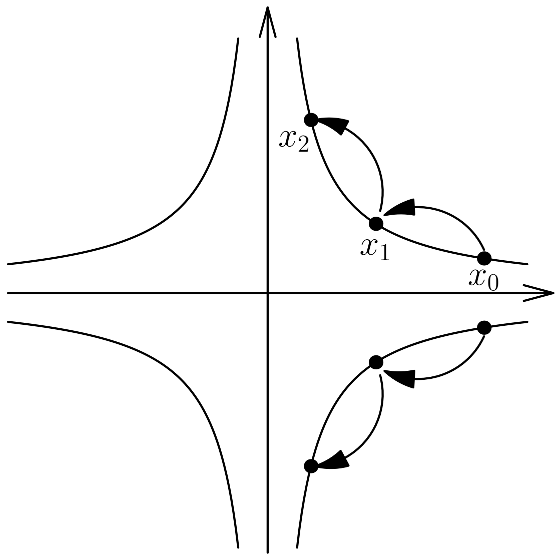



Figure 3.22: Saddles for discrete time equations (a) $0 < \lambda_1 < 1 < \lambda_2$; (b) $-1 < \lambda_1 < 0$, $\lambda_2 > 1$.

where $\lambda = \sqrt{\rho^2 + \omega^2}$ and $\theta = \tan^{-1}(\omega/\rho)$. If we define $z_n = x_n + iy_n$, then in complex notation

$$
\begin{aligned}
z_{n+1} = x_{n+1} + iy_{n+1} = \lambda\big[(x_n \cos\theta - y_n \sin\theta) + i(x_n \sin\theta + y_n \cos\theta)\big] \\
= \lambda\big(\cos\theta + i\sin\theta\big)(x_n + iy_n) = \lambda e^{i\theta} z_n.
\end{aligned}
$$

Therefore, the solution can be written as $z_n = \lambda^n e^{in\theta} z_0$, or equivalently

$$
x_n = \lambda^n(x_0 \cos n\theta - y_0 \sin n\theta), \quad y_n = \lambda^n(x_0 \sin n\theta + y_0 \cos n\theta).
$$

Therefore, the solution $(x_n, y_n)$ converges to the origin if and only if $\lambda = \sqrt{\rho^2 + \omega^2} < 1$.