MATH 44041/64041
Applied Dynamical Systems

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Recommended Reading:

(1) Morris W. Hirsch, Stephen Smale, and Robert L. Devaney. *Differential equations, dynamical systems, and an introduction to chaos.* Elsevier/Academic Press, Amsterdam, 3rd edition, 2013.

(2) James D. Meiss. *Differential dynamical systems*, volume 14 of *Mathematical Modeling and Computation.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

(3) Steven H. Strogatz. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering.* Westview press, 2014.

(4) Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos.* Springer-Verlag, New York, second edition, 2003.

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1 Introduction

1.1 Problems modelled with differential or difference equations

Applied mathematicians create mathematical models to describe the world. These may involve physics (mechanics), chemistry (reaction kinetics), economics (stock movements, supply and demand), social sciences (voter preferences, opinion formation) or any number of different disciplines and problems. The common thread though is that the model is only useful if it can be used to obtain more insights into the problem being addressed. The methods that can be brought to bear depend on the nature of the model.

Models used to simulate and predict weather or climate could be very complicated, because various processes like heat transfer (both vertically and horizontally) are coupled together on the surfaces of land, ocean and ice. For the fantastically detailed climate models used to assess the probability of climate change the techniques are essentially computational, but mathematics is important in the design of the schemes and the analysis of the data. Climate scientists will also use much cruder models to provide insights into the relative importance of different effects. These models are designed so that more detailed mathematical analysis is possible, and longer, more varied computer simulation as well because the time spent on the computation is so much smaller.

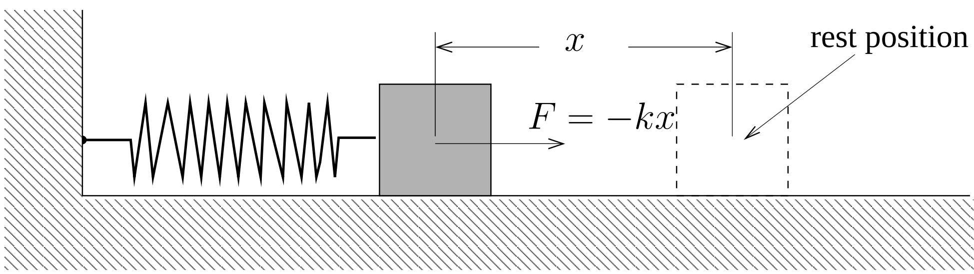

Figure 1.1: A harmonic oscillator only under the spring force $F = -kx$.

The aim of this course is to describe some of the mathematical techniques that can be used to analyse differential or difference equations that arise frequently in models. Differential equations are used to describe how quantities vary in time (or space). If there is only one independent variable then the model is an ordinary differential equation (ODE) such as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{1.1}$$

with solutions $x(t)$ that is a function of the continuous, independent variable t and the initial conditions (if they are specified). This equation describes the motion of an object under the sole force of spring force (see Figure 1.1), governed by the Newton's equation $m\ddot{x} = F = -kx$. It is sometimes useful to consider time as a discrete variable, leading to difference equations such as the logistic equation

$$x_{n+1} = \mu x_n(1 - x_n) \quad \text{with} \;\; \mu \in [0, 4]. \tag{1.2}$$

This generates a sequence $\{x_0, x_1, x_2, \dots\}$ rather than a function of a continuous variable. We assume you are familiar with basic linear differential equations and difference equations.

There are two features that may be new to you (and will be our focus later) in this course: **nonlinearity** and **parameter variation**.

Nonlinearity refers to the existence of terms like x^2 in the equation – terms that are not linear in the *dependent* variable you are seeking — for example, the logistic equation (1.2) is nonlinear whilst equations (1.1) and are linear in x and u respectively. In general, nonlinear equations cannot be solved in terms of simple functions, and new techniques are needed to obtain information about solution.

In many models these are **parameters** — quantities which are constant in any single realisation of the experiment, but which can be changed (like the interest rate set by Bank of England to regulate the economy). In fluid mechanics an example is the Reynolds number of a flow, in chemistry reaction rates depend on ambient temperatures, in social sciences behaviour may be influenced by the average number of friends a person has (and in epidemiology the average number of contacts). Often these parameters are not known accurately and so it is important to know how sensitive any conclusions are to parameter variation. This is described by *bifurcation theory*: the study of how quantitative changes occur as parameters are varied. The quantity μ in the logistic equation is an example of a parameter. 'Tipping points' are another.

Finally, nonlinearity can lead to behaviour that is more complicated than the obvious static and periodic solutions (or more general quasi-periodic solutions). This is called *chaos*, and one of the interesting features of chaos is that it has its own version of bifurcation theory — there are a number of common *routes to chaos* describing how chaotic solutions are created as parameters change. We will discuss some of these too.

1.2 What is this course about?

In this course, we will be focus on *qualitative properties* of continuous and discrete dynamical systems, complementing other common methods, like explicit solutions and numerical approximations. Explicit solutions, even available in certain cases, may not be useful. For example, the general solution of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1 - x^2 + y^2}{2}$$

with initial condition $x(0) = x_0, y(0) = y_0$ is

$$\begin{aligned} x(t) &= \frac{2x_0}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t}, \\ y(t) &= \frac{2y_0 \cos t + (1 - x_0^2 - y_0^2)\sin t}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t}. \end{aligned}$$

Can you get any information from this explicit solution, without plotting any sample trajectories? Numerical approximations may also not be so effective for high dimensional systems or long time behaviours.

Before considering complicated nonlinear systems, we start with a few basic notations and concepts in the next section.

2 Notation and basic concepts

2.1 Ordinary differential equations (ODEs)

We only consider coupled *first order* equations: *autonomous*(time-independent) equations

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,\ f : \mathbb{R}^n \mapsto \mathbb{R}^n,$$

or occasionally *non-autonomous* (time-dependent) ones

$$\dot{x} = f(x, t), \qquad x \in \mathbb{R}^n,\ f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n.$$

Here the n-dimensional Euclidean space $\mathbb{R}^n$ is called the *phase space* of the system.

Remark (Conventions about notations).

(a) In the rest of the course, we use variables like x for both vector and scalar, and there is usually no confusion. For instance, x in the system $\dot{x} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x$ is a column vector of length two, but x in the system

$$\dot{x} = x - y, \quad \dot{y} = -x + y$$

is a scalar.

(b) Explicit time dependence is also omitted, that is, we write x instead of $x(t)$, and similarly $\dot{x}, \ddot{x}$ for the time derivatives $\frac{\mathrm{d}}{\mathrm{d}t}x(t)$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$.

(c) The solution to the system $\dot{x} = f(x)$ is usually written as x or $x(t)$, and sometime $x(x_0, t)$ or $\varphi_t(x_0)$, if the dependence on the initial condition x_0 is emphasized.

There is no need to consider higher order equations, because they can always be converted into first order systems by introducing new variables for the derivatives, as in the following example.

Example 2.1 (Coupled first order equations)**.** Take the simple harmonic oscillator (1.1),

$$\ddot{x} + \omega^2 x = 0. \tag{2.1}$$

This is a second order equation, and can be recast in the form of (2.1) by setting $y = \dot{x}$ (hence $\dot{y} = \ddot{x} = -\omega^2 x$). That is, we get the coupled first order equations

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that this is an example of a *linear* system of differential equations, the general form of which is

$$\dot{v} = Av. \qquad v \in \mathbb{R}^n$$

where A is a $n \times n$ matrix (possibly time dependent).

Exercise. Change the fourth order equation $\frac{\mathrm{d}^4}{\mathrm{d}t^4}x - \omega^4 x = 0$ describing the vibration of a beam into a system of first order equations, by introducing $y = \dot{x}, z = \ddot{x}, w = \dddot{x}$. What does the coefficient matrix look like? How about the equivalent first order system for the n-th order autonomous ODE

$$x^{(n)} = F\big(x, x', \cdots, x^{(n-1)}\big),$$

by introducing the variables $x_k = x^{(k)}$, $k = 0, 1, 2, \cdots, n-1$ for the k-th order derivatives.

We can always consider autonomous equations (with no explicit dependence on "time"), because non-autonomous system like $\dot{x} = f(x,t)$ can be converted into autonomous system by introducing a new independent variable τ as "time", while taking t as dependent variable. That is, $\dot{x} = f(x,t)$ is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x = f(x,t), \quad \frac{\mathrm{d}}{\mathrm{d}\tau}t = 1,$$

which is an autonomous system for $\tilde{x} = (x,t)$ that depends on τ.

Example 2.2 (The simplest scalar linear equation)**.** The differential equation $\dot{x} = ax$ for $x \in \mathbb{R}$ and with initial condition $x(0) = x_0$ is simple but illustrates features of stability and instability to which we will return. If $x_0 = 0$ then $\dot{x} = 0$ and so $x(t) = 0$ for all time (it is a *stationary point*). If $x_0 \neq 0$, it can be solved using separation of variables, that is,

$$\int_{x_0}^{x} \frac{\mathrm{d}x}{x} = \int_0^t a\mathrm{d}t,$$

which gives $x = x_0 e^{at}$. Alternatively, we can use the *integrating factor* e^{-at}. Taking the derivative of $e^{-at}x$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-at}x) = -ae^{-at}x + e^{-at}\dot{x} = e^{-at}(\dot{x} - ax) = 0.$$

Therefore, $e^{-at}x$ is a constant and equal its value at $t = 0$. That is $e^{-at}x = x_0$, or $x = e^{at}x_0$. The behaviour of solutions depends on the sign of a (the real part of a if it is complex):

$$\text{if } a < 0 \text{ then } |x(t)| \to 0; \qquad \text{if } a > 0 \text{ then } |x(t)| \to \infty.$$

A radioactive material contains unstable nuclei whose atomic nucleus loses energy and decays into another nuclide. Let N_A be the number [1] of atoms in a sample, then N_A is usually governed by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}N_A = -\lambda_A N_A$$

where λ_A is the decay constant. The solution is $N_A(t) = N_A(0)e^{-\lambda_A t}$. The time $T_{1/2} = \frac{\ln 2}{\lambda_A}$ is called *half-life*, is the time taken for the radioactive substance to decay to half of the initial value, i.e., $N_A(T_{1/2}) = N_A(0)/2$.

[1] This number is so large in practice that it can be treated as a continuous quantity to be differentiated.

Example 2.3 (Chain of two radioactive decays)**.** If one nuclide A decays into B by one process, and then B decays into C by a second process, then the amounts of A and B are governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}N_A = -\lambda_A N_A, \qquad \frac{\mathrm{d}}{\mathrm{d}t}N_B = -\lambda_B N_B + \lambda_A N_A,$$

with the initial condition $N_B(0) = 0$ (no B at the very beginning). From the solution $N_A(t) = N_A(0)e^{-\lambda_A t}$, the second equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}N_B(t) = -\lambda_B N_B(t) + \lambda_A N_A(0)e^{-\lambda_A t}.$$

If $\lambda_B \neq \lambda_A$, then this ODE can be integrated with the integrating factor $e^{\lambda_B t}$ to give

$$\lambda_B(t) = \frac{N_A(0)\lambda_A}{\lambda_B - \lambda_A}\big(e^{-\lambda_A t} - e^{-\lambda_B t}\big).$$

Exercise. (1) Find the time T when $N_B(t)$ reaches its maximum; (2) Find the solution when $\lambda_B = \lambda_A$.

Example 2.4 (Linear matrix equations)**.** The previous system can be written as

$$\dot{x} = Ax \tag{2.2}$$

with $x \in \mathbb{R}^n$, and where A is an $n \times n$ constant matrix. Solutions can be written as

$$x(t) = e^{tA}\, x_0 \tag{2.3}$$

where the exponential matrix is defined by (exactly the same as in the scalar case)

$$e^B \;=\; I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}B^n,$$

where I is the identity matrix. With this definition of matrix exponential, the expression (2.3) is a solution to the linear matrix equation (2.2) can be proved by differentiating term by term. In practice, the matrix exponential e^B is not calculated from above series expansion, but by transforming B into Jordan blocks, using eigenvectors of B. If $B = S\Lambda S^{-1}$, where the columns of S are the (generalised) eigenvalues of B, and Λ consists of Jordan blocks:

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_m \end{pmatrix}, \qquad \Lambda_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix}.$$

Then $e^B = Se^{\Lambda}S^{-1}$, while e^{Λ} can be computed easily. In general to find e^B, it is easier to find the eigenvectors and eigenvalues (or equivalently the decomposition $B = S\Lambda S^{-1}$) than to calculate the series with powers B^n (but there are exceptions as in the following example).

Example 2.5. If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, using the fact that $A^{4n} = I, A^{4n+1} = A, A^{4n+2} = -I, A^{4n+3} = -A$ (n is an integer) and the above definition for matrix exponential, we get

$$\exp(tA) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Exercise. What is $\exp(tA)$ for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

Review on different ways to solve differential equations:

(i) **Linear ODEs with constant coefficients**:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x(t) + \cdots + a_1\frac{\mathrm{d}}{\mathrm{d}t}x(t) + a_0 x(t) = 0.$$

Looking for solution of the form $x(t) = e^{\omega t}$, where ω are the roots of the n-th degree polynomial

$$\omega^n + a_{n-1}\omega^{n-1} + a_1\omega + a_0 = 0.$$

(ii) **Linear first order scalar equation $\dot{x} = a(t)x + b(t)$:** Multiply both sides by the *integrating factor* $\exp\left(-\int^t a(\tau)d\tau\right)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x\exp\left(-\int^t a(\tau)d\tau\right)\right] = [\dot{x} - a(t)x]\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right) = b(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right).$$

followed by integrating on both sides,

$$x(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right) = C + \int b(t)\exp\left(-\int^t a(\tau)\mathrm{d}\tau\right)\mathrm{d}t.$$

(iii) **Separable first order equation $\dot{x} = f(x)g(t)$:** Integrate $\dfrac{\mathrm{d}x}{f(x)} = g(t)\mathrm{d}t$ to get

$$\int^x \frac{\mathrm{d}x}{f(x)} = \int^t g(t)\mathrm{d}t,$$

where the integration constant is determined by the initial condition (if it is given).

(iv) **First order homogeneous ODEs $\dot{x} = f(t,x)$, where $f(\lambda x, \lambda t) = f(x,t)$ for any λ:** The trick is to introduce $z = x/t$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \frac{\mathrm{d}}{\mathrm{d}t}(zt) = z + t\frac{\mathrm{d}z}{\mathrm{d}t}$$

and $f(x,t) = f(zt,t) = f(z,1)$, the original ODE becomes $z + t\dot{z} = f(z,1)$, which is separable, and the solution is given by

$$\int \frac{\mathrm{d}t}{t} = \int \frac{\mathrm{d}z}{f(z,1) - z}.$$

System of equations, especially nonlinear ones, are much more difficult to solve analytically, if not impossible. Nevertheless, we can still have a good understanding of the qualitative properties, using different techniques that will be developed in the rest of the course.

2.2 Trajectories, phase portrait and flow on the phase space

In many situations, although explicit solutions of the underlying equations may not be available, qualitative properties and long time behaviours can still be obtained using various techniques. For example, we can understand solutions of the logistic ODE

$$\dot{x} = x(1-x)$$

with different initial conditions $x(0)$. If $x(0) < 0$, then $x(t)$ decreases, and $x(t) \to -\infty$ as $t \to \infty$. If $x(0) \in (0,1)$, then $x(t)$ increases to 1 and finally if $x(0) > 1$, $x(t)$ decreases to 1. In general, for the one dimension equation $\dot{x} = f(x)$, although we can get the solution from

$$\int_{x(0)}^{x(t)} \frac{\mathrm{d}x}{f(x)} = t,$$

the qualitative properties can be understood better using a *phase portrait* as Figure 2.1: x increases on regions where $f(x) > 0$ and decreases where $f(x) < 0$.

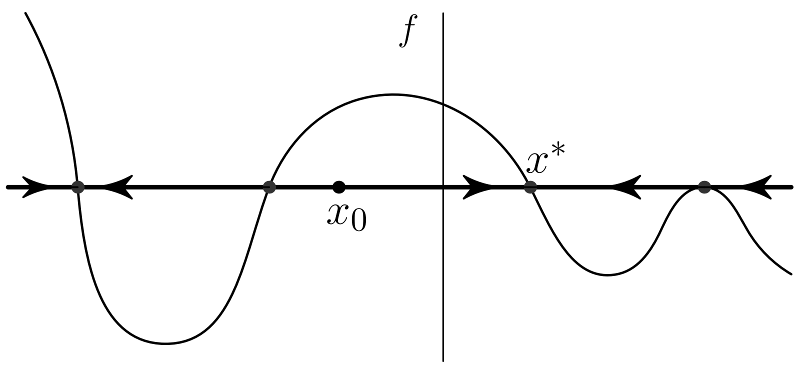

Figure 2.1: Phase portrait of the one dimensional autonomous equation $\dot{x} = f(x)$.

This picture can be extended to higher dimensions. Consider the equation $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. If we plot the trajectory $\{x(t) \mid t_1 \le t \le t_2\}$ in $\mathbb{R}^n$ for some time t_1 and t_2, then $f(x(t))$ is exactly the tangent vector of $x(t)$ (the definition of the ODE $\dot{x} = f(x)$!). In other words, once we have the *vector field* $f(x)$ at any points x, then we can "integrate" along the vector field to get the solution trajectory, as in Figure 2.2. A sketch of the different trajectories in phase space is called a *phase portrait*. Indicate the direction of time on phase portraits by an arrow denoting the direction of increasing time along the trajectory. In some cases, some trajectories can be obtained explicitly by solving ODEs with time t eliminated; otherwise, general behaviour of the underlying system can be inferred by "connecting" the vector field given by $f(x)$.

Example 2.6 (Equations governed by trajectories)**.** Consider the system

$$\dot{x} = -y, \quad \dot{y} = x.$$

The trajectory is governed by the different equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = -\frac{x}{y},$$

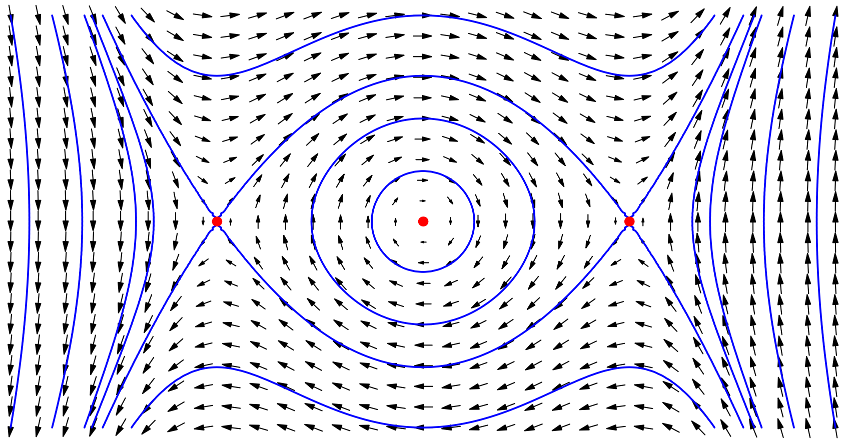

Figure 2.2: Vector fields and phase portrait for the system $\dot{x} = y, \dot{y} = -x + x^3$.

which is separable. Rewriting this ODE as $y\mathrm{d}y + x\mathrm{d}x = 0$ and integrating both sides, we get

$$x^2 + y^2 = C$$

for some constant $C > 0$.

Example 2.7 (Newtonian dynamics in one dimension)**.** Consider the Newtonian dynamics $m\ddot{x} = -U'(x)$ in one dimension (so x is a scalar, m is the mass and U is called the *potential*). By introducing the momentum $p = m\dot{x}$, then the original second order scalar ODE is equivalent to the first order system

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -U'(x).$$

The trajectory, governed by the ODE $\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{mU'(x)}{p}$ is separable, is

$$\frac{p^2}{2m} + U(x) = E$$

for some constant E, called the total *energy*.

Example 2.8. For the ODE $\dot{x} = y, \dot{y} = -x + x^3$, the ODE governing the trajectory is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x + x^3}{y}.$$

Therefore the trajectories are $x^2 + y^2 - x^4/2 = C$ for some constant C (not necessarily positive).

Remark. For two dimensional system $\dot{x} = f(x, y), \dot{y} = g(x, y)$, the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$ can not always be solved explicitly. For instance, if the system in the previous example is changed to $\dot{x} = y + x, \dot{y} = -x + x^3$, there seems no expressions for general trajectories. But whenever there is a solution that written in the form $F(x, y) = C$, the function $F(x, y)$ is called a *conserved quantity* (because the time derivative $\frac{d}{dt}F(x, y)$ is zero), containing important information about the underlying system.

Remark. In most cases, it is easier to write the trajectories *implicitly* as $x^2 + y^2 = C$ or $x^2 + y^2 - x^4/2 = C$ in previous two examples. There is no need to write y as a function of x or x as a function of y.

Semi-group property for autonomous ODEs

Another way of representing solutions is via the *flow*: $x(t) = \varphi_t(x_0)$ represents the solution to $\dot{x} = f(x)$ at time t with initial condition x_0 at $t = 0$, i.e. $\varphi_0(x_0) = x_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t 2(x_0) = f(\varphi_t(x_0)).$$

For example, the solution to the system of ODEs

$$\dot{x} = -x + y, \quad \dot{y} = -y$$

with initial condition (x_0, y_0) is given by

$$\varphi_t(x_0, y_0) = (x_0 e^{-t} + te^{-t}y_0, e^{-t}y_0).$$

For autonomous equations $\dot{x} = f(x)$, where f has no explicit dependence on t, the solution $\varphi_t(x)$ satisfies the semi-group property,

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x)) = \varphi_s(\varphi_t(x)).$$

This fact can be verified by the uniqueness of the solution to the system $\dot{x} = f(x)$, by defining two functions $\psi_1(t) = \varphi_{t+s}(x), \psi_2(t) = \varphi_t(\varphi_s(x))$. Then $\psi_1(t)$ is a solution to $\dot{x} = f(x)$ with initial condition $\psi_1(0) = \varphi_s(x)$ and $\psi_2(t)$ is also a solution to $\dot{x} = f(x)$ with initial condition $\psi_2(0) = \varphi_0(\varphi_s(x)) = \varphi_s(x)$. Since $\psi_1(0) = \psi_2(0)$, by the uniqueness of solutions to ODEs, $\psi_1(t) = \psi_2(t)$, or $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$. Similarly, we can show $\varphi_{t+s}(x) = \varphi_s(\varphi_t(x))$.

Example 2.9. The solution to the ODE $\dot{x} = x^2, x(0) = x_0$ satisfies the semi-group property. In fact, this is an separable ODE. Integrating both sides of $x^{-2}\mathrm{d}x = dt$, we get

$$t = \int_0^t \mathrm{d}t = \int_{x_0}^{x(t)} \frac{\mathrm{d}x}{x^2} = \frac{1}{x_0} - \frac{1}{x(t)}.$$

That is $\varphi_t(x_0) = \frac{x_0}{1-tx_0}$ and $\varphi_s(\varphi_t(x_0)) = \varphi_s\left(\frac{x_0}{1-tx_0}\right) = \frac{\frac{x_0}{1-tx_0}}{1-s\frac{x_0}{1-tx_0}} = \frac{x_0}{1-(t+s)x_0} = \varphi_{t+s}(x_0)$.

Remark. The reason we use *semi-group* instead of *group* here is that some dynamical systems can not be defined backward in time, or lose the uniqueness of solution when solving backward in time (common for infinite dimensional systems, like partial differential equations).

Remark. The solution of any *autonomous* system always satisfies the semi-group property (the law of dynamics does not dependent on "time"); on the other hand, if a function $\varphi_t(x_0)$ satisfies the semi-group property, then it is the solution to the first order autonomous system $\dot{x} = f(x), x(0) = x_0$. The function f, or the "law of dynamics" can be actually determined by writing $\frac{d}{dt}\varphi_t(x_0)$ as a function $f(\varphi_t(x_0))$. For instant, if $\varphi_t(x_0) = x_0/(1 - tx_0)$, then $\frac{d}{dt}\varphi_t(x_0) = x_0^2/(1-tx_0)^2 = (\varphi_t(x_0))^2 = f(\varphi_t(x_0))$ with $f(x) = x^2$, the same as in Example 2.9 (there is no explicit t dependence). Alternatively, $f(x)$ can be determined at the initial time (the law can be inferred from any instance of time). That is, $f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\big|_{t=0}$.

2.3 Special solutions: fixed points and periodic orbits

Special solutions, if they exists, usually give a lot of information about the general behaviour of the underlying system. There are two obvious special solutions for $\dot{x} = f(x)$ arising in practice:

Stationary (or fixed) points: A *stationary point* x^* satisfies

$$x(t) \equiv x^*, \tag{2.4}$$

i.e. the trajectory is a single point and the solution does not change in time. Thus

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = f\big(x(t)\big) = f(x^*),$$

and stationary points can be found by solving the algebraic equation $f(x^*) = 0$.

Periodic Orbits: if there exists $T > 0$ such that

$$x(t+T) = x(t) \ \text{ for all } t \in \mathbb{R}.$$

then the trajectory is called a periodic orbit and T is called the period of the periodic orbit. Note that kT is also a period for any positive integer k because $x(t+kT) = x(t)$, and sometimes T is referred as the *minimal period*). A *periodic orbit* with $T = 0$, which is not allowed in the definition, would be a stationary point. Periodic orbits are much harder to find, and they form *closed curves* in phase space.

Example 2.10 (Fixed points of linear constant coefficient ODEs)**.** If A is a non-singular $n \times n$ matrix, then the only fixed point is the origin. In other words, the only solution to $Ax = 0$ is $x = 0$.

Exercise. What if the coefficient matrix A is singular as in

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the solution starting from (x_0, y_0). What are the fixed points?

Example 2.11 (Fixed points of potential dynamics)**.** Consider the Newton's equation $m\ddot{x} = -\nabla U(x)$ in n-dimensional space (the force is derived from the potential U), which is equivalent to the first order system of $2n$ equations ($p = m\dot{x}$ is the linear momentum):

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = -\nabla U(x).$$

Then any fixed point takes the form $(x^*, p^* = 0)$, where $\nabla U(x^*) = 0$. For those who took courses in mechanics, the fixed point is stable if x^* at the local minimum (bottom of the potential well), and unstable if x^* is at a saddle point.

Example 2.12 (Harmonic oscillation)**.** The simplest example of periodic phenomenon is the motion of a harmonic oscillator, $\ddot{x} + \omega^2 x = 0$, or the equivalent first order system

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x.$$

The only fixed point is the origin, but there are many periodic orbits around the origin. In fact, the solution can be written as

$$x(t) = A\cos\omega t + B\sin\omega t.$$

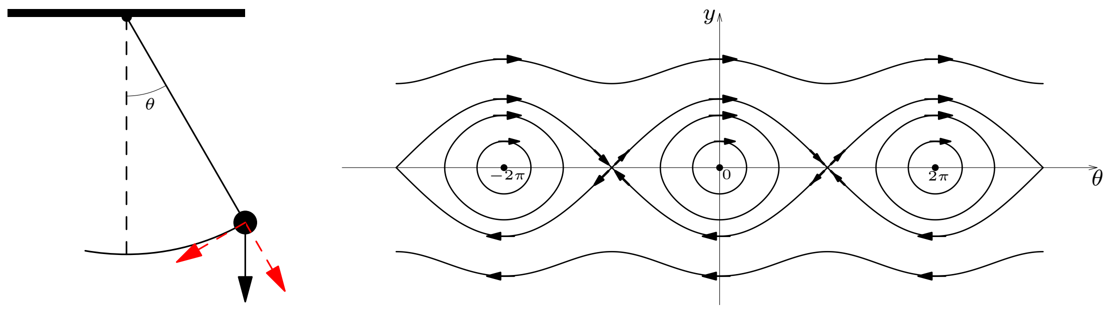

Figure 2.3: The simple pendulum and associated phase portrait.

Example 2.13 (Simple Pendulum)**.** Consider the pendulum in Figure 2.3. By taking components of the force in the radial direction, the equation of motion is

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

or the first order system (by introducing $y = \dot{\theta}$)

$$\dot{\theta} = y, \qquad \dot{y} = -\frac{g}{\ell}\sin\theta.$$

So phase space is $\mathbb{R}^2$, or more precisely the cylinder $\mathbb{T} \times \mathbb{R}$ with $\theta \in [0, 2\pi)$ (here θ is taken modulo 2π). The solution can not be represented using elementary functions, but can be given in terms of more special ones called *elliptic functions*.

Stationary points are given by solving $\dot{\theta} = \dot{y} = 0$, i.e. $y = 0$ and $\sin\theta = 0$, so the stationary points are (see Figure 2.3)

$$(k\pi, 0) \quad k \in \mathbb{Z}.$$

The only fixed point is the origin, but there are many periodic orbits around the origin.

The simple pendulum equation has special properties that make it easier to sketch the phase portrait than for more general systems: the energy (also called *Hamiltonian*)

$$E = \frac{1}{2}y^2 - \frac{g}{\ell}\cos\theta$$

is constant on solutions, which is determined from the initial condition $(\theta_0, y_0 = \dot{\theta}(0))$. This can be seen by differentiating both sides with respect to time (using the chain rule on the right hand side):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = y\,\dot{y} + \dot{\theta}\frac{g}{\ell}\sin\theta = \frac{g}{\ell}\left(-y\ \sin\theta + y\ \sin\theta\right) = 0.$$

Example 2.14 (Prey-predator system)**.** Let x and y be the population number of prey (for example, rabbits) and predator (for example, foxes), then the simplest system of ODEs is

$$\dot{x} = (A - By)x, \qquad \dot{y} = (Cx - D)y, \tag{2.5}$$

where A, B, C, D are all positive constants. The fixed points are

$$(0,0), \qquad \left(\frac{D}{C}, \frac{A}{B}\right).$$

2.4 Invariant sets

For the ODE $\dot{x} = f(x)$, a set $\mathcal{S} \subseteq \mathbb{R}^n$ is called *invariant*, if $x_0 \in \mathcal{S}$ implies the solution $x(t) = \varphi_t(x_0) \in \mathcal{S}$ with initial condition $x(0) = \varphi_0(x_0) = x_0$ for all $t \geq 0$. The basic idea behind invariant sets is: if you start in the set, you stay in the set. Common invariant sets include:

(1) Single/multiple stationary points

(2) Periodic orbits

(3) Trajectory passing one or more specific points

$$\mathcal{S}_+(x_0) = \{\varphi_t(x_0) \mid t \geq 0\} \qquad \text{or} \qquad \mathcal{S}(x_0) = \{\varphi_t(x_0) \mid t \in \mathbb{R}\}.$$

Example 2.15. The unit circle $x^2 + y^2 = 1$ is invariant for the system

$$\dot{x} = -x + y + x(x^2 + y^2), \qquad \dot{y} = -x - y + y(x^2 + y^2).$$

In other words, if (x_0, y_0) is on the unit circle, then the solution is also on the unit circle for any time $t > 0$. Therefore, we only need to show that $x^2 + y^2$ does not change (always unit), for all time. Taking the time derivative of $x^2 + y^2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 2x\big(-x + y + x(x^2 + y^2)\big)& \\ + 2y\big(-x - y + y(x^2 + y^2)\big) = 2(x^2 + y^2)(x^2 + y^2 - 1) = 0.&\end{aligned}$$

That is, $x^2 + y^2$ does not change in time if (x, y) is on the circle, and $x^2 + y^2 = 1$ for all time.

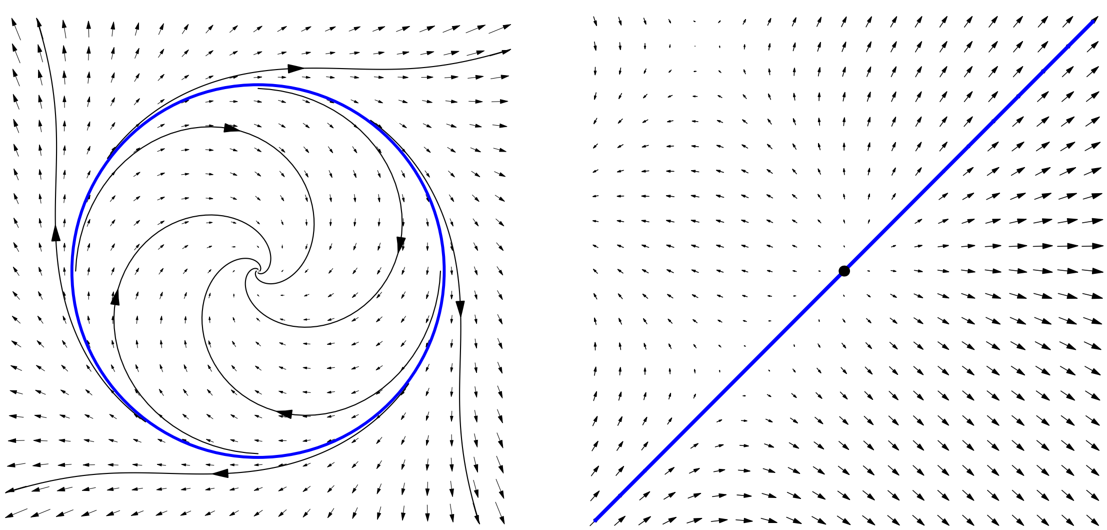

Figure 2.4: Invariant circle for Example 2.15 and invariant straight line for Example 2.17.

This example shows an important fact: a set $\mathcal{S} = \{x \mid G(x) = 0\}$ is *invariant* iff

$$\frac{\mathrm{d}G}{\mathrm{d}t} = f \cdot \nabla G = 0 \qquad \text{on} \quad G(x) = 0.$$

Geometrically, ∇G is the normal to the curve $\mathcal{S}$, and $f \cdot \nabla G = 0$ means that the vector field defining the ODE is orthogonal to the normal.

Example 2.16 (Prey-predator system)**.** The two coordinate axis $\mathcal{S}_x = \{(x,0)\}, \mathcal{S}_y = \{(0,y)\}$ are two invariant sets of the system

$$\dot{x} = (A - By)x, \qquad \dot{y} = (Cx - D)y.$$

If the initial condition (x_0, y_0) is on the x-axis, then $y_0 = 0$, then the solution $(x(t), y(t))$ stays on the x-axis, or $y(t) \equiv 0$, because $\dot{y} \equiv 0$. Alternatively, from the second ODE $\dot{y}(t) = (Cx(t) - D)y(t)$, we can "solve" $y(t)$ (assuming $x(t)$ is known, this is a linear ODE)

$$y(t) = y_0 \exp\left[\int_0^t (Cx(\tau) - D)\mathrm{d}\tau\right] = 0.$$

Therefore, the x-axis is invariant. Similarly, we can show y-axis is invariant.

Example 2.17. We can show that the line $y = 2x$ is invariant under the system

$$\dot{x} = \frac{5}{2}x - \frac{1}{2}y + 2x^2 + \frac{1}{2}y^2, \qquad \dot{y} = -x + 2y + 4xy.$$

Geometrically, the line $y = 2x$ is a trajectory on the phase portrait.

Define $G(x,y) = y - 2x$, so the line is $\mathcal{S} = \{(x,y) \mid G(x,y) = 0\}$. We can look at the evolution of the function G under the system,

$$\begin{aligned}\dot{G}(x,y) &= \dot{y} - 2\dot{x}\\ &= (-x + 2y + 4xy) - (5x - y + 4x^2 + y^2)\\ &= -6x + 3y - 4x^2 + 4xy - y^2\end{aligned}$$

If $(x,y) \in \mathcal{S}$, $G = 0$ and $y = 2x$, which implies that

$$\dot{G}|_{G=0} = -6x + 6x - 4x^2 + 8x^2 - 4x^2 = 0.$$

In physics, the invariance of a set is generally related to the conservation of some quantities, as shown in the following three examples.

Example 2.18 (Conservation of energy for Newtonian potential dynamics)**.** If the force F of an particle with mass m is derived from a potential (gravitational potential or electric potential), that is $F(x) = \nabla U(x)$ for some U, then the Newton's equation becomes $\ddot{x} = F(x) = -\nabla U(x)$. Introduce the (linear) momentum $p = m\dot{x}$, then the dynamics is governed by the equivalent first order system

$$\dot{x} = p/m, \qquad \dot{p} = -\nabla U(x).$$

Then the total energy (also call *Hamiltonian*) $E(x,p) = \dfrac{p^2}{2m} + U(x)$ is conserved, and the dynamics is on the *constant energy surface.*

Example 2.19. We can show that the (open) unit disk $\{(x,y) \mid x^2 + y^2 < 1\}$ is invariant for the system

$$\dot{x} = -x + y, \quad \dot{y} = -x - y.2$$

By the definition, we need to show that if the initial condition (x_0, y_0) is on the unit disk (i.e. $x_0^2 + y_0^2 < 1$), then $x(t)^2 + y(t)^2 < 1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2+y^2\right) = 2x\dot{x} + 2y\dot{y} = 2x(-x+y) + 2y(-x-y) = -2(x^2+y^2) \le 0,$$

the quantity $x(t)^2 + y(t)^2$ is non-increasing. In other words, $x(t)^2 + y(t)^2 \le x_0^2 + y_0^2 < 1$. Therefore, the point $(x(t), y(t))$ stays22 on the unit disk.

Example 2.20 (Bounding functions)**.** The previous example can be generalised into the concept of bounding functions. Let $V(x) = c$ be a set of nested regions with c increasing outwards, that is $\{x \in \mathbb{R}^n \mid V(x) \le c_1\} \subset \{x \in \mathbb{R}^n \mid V(x) \le c_2\}$ for $c_1 \le c_2$. If

$$f \cdot \nabla V < 0 \qquad \text{on } V(x) = c$$

for some c, then the set $\{x \in \mathbb{R}^n | V(x) \le c\}$ is invariant. The idea of the proof is very simple (we will cover it in more detail later): if $f \cdot \nabla V < 0$ then f must point inwards along the surface and so no solutions can leave the region $V < c$ across the surface.

2.5 Existence and uniqueness

We have been assuming the existence of solutions of dynamical systems without comment. However this is not necessarily straightforward and needs to be examined in more depth. As the next set of examples show, solutions for ODEs may be difficult to pin down!

Different phenomena in ODEs: We will give a sequence of examples showing how complications can arise in ODEs.

Example 2.21 (Non-uniqueness with continuous right hand side)**.** Consider the ODE $\dot{x} = \sqrt{|x|}, \quad x_0 = 0$. By observation $x(t) = 0$ is a solution (a stationary point). On the other hand, using separation of variables, we get

$$\int^x \frac{dx}{\sqrt{|x|}} = \int^t dt.$$

If $x \ge 0$, both sides of above equation become $2(\sqrt{x} - \sqrt{x_0}) = t$. Therefore $x(t) = t^2/4$ is a different solution other than the trivial one $x(t) \equiv 0$! Even worse, we have a family of functions $x_\tau(t)$ for $\tau \ge 0$, defined by

$$x_\tau(t) = \begin{cases} 0, & \text{if } 0 \le t \le \tau, \\ (t-\tau)^2/4, & \text{if } t > \tau, \end{cases}$$

as can be verified by direct substitution (check it!). The main issue responsible for the non-uniqueness here is that $f(x) = \sqrt{|x|}$ is *not* Liptchitz continuous.

Example 2.22 (Finite time blow up)**.** Consider the differential equation

$$\dot{x} = x^2$$

with solutions $\int \frac{dx}{x^2} = t$ or $x = \frac{x_0}{1 - x_0 t}$. Thus if $x_0 > 0$ then solutions tend to infinity as $t \to x_0^{-1}$.

These examples show that we need a better understanding of existence of solutions. To show the existence, we first convert the ODE $\dot{x} = f(x,t),\ x(0) = x_0$ into an integral equation

$$x(x_0, t) = x_0 + \int_0^t f(x(s), s)ds, \tag{2.6}$$

which can be verified by differentiating and using the Fundamental Theorem of Calculus. Of course, if we do not know $x(s)$, $0 \le s < t$, then this does not help as we cannot evaluate the integral. Instead we consider the iteration

$$x^{(n+1)}(t) = T[x^{(n)}](t), \qquad \text{where } \ T[x](t) = x_0 + \int_0^t f(x(s), s)ds \tag{2.7}$$

with the initial condition $x^{(0)}(t) = x_0$. If the sequence of functions $\{x^{(n)}(t)\}$ converges to some function $\bar{x}(t)$, then taking the limit of both sides of (2.7), we get $\bar{x}(T) = T[\bar{x}](t)$, or $\bar{x}$ is a fixed point of the operator T. Taking derivative of both sides of $\bar{x}(T) = T[\bar{x}](t)$, we can show that $\bar{x}$ is a solution of the ODE $\dot{x} = f(x,t)$ with initial condition $\bar{x}(0) = x_0$. This is called *Picard Iteration*, and so if we can show that T defined in (2.7) is a contraction mapping then we have an existence theorem. On the assumption that this can be done Picard iteration also provides a way of constructing approximate solutions locally (See Figure 2.5).

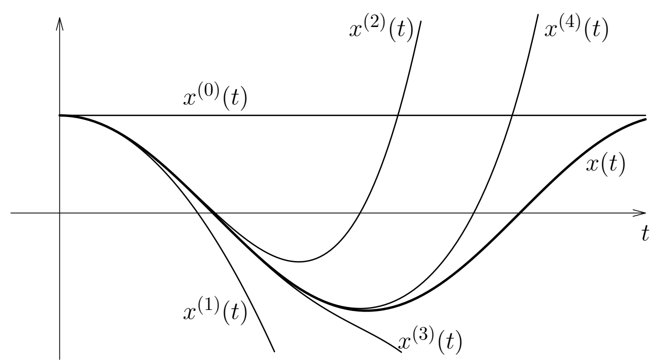

Figure 2.5: The sequence of function $x^{(n)}(t)$ converges to the exact solution $x(t)$.

Example 2.23. For the ODE $\dot{x} = ax$ with initial condition $x(0) = 1$. The Picard iteration is

$$x^{(n+1)}(t) = 1 + \int_0^t ax^{(n)}(s)ds$$

with $x^{(0)}(t) = x_0 = 1$. Therefore,

$$\begin{aligned}
x^{(1)}(t) &= x_0 + \int_0^t ax^{(0)}(s)ds = 1 + \int_0^t ads = 1 + at,\\
x^{(2)}(t) &= x_0 + \int_0^t ax^{(1)}(s)ds = 1 + \int_0^t a(1+as)ds = 1 + at + \frac{a^2}{2}t^2,\\
x^{(3)}(t) &= x_0 + \int_0^t ax^{(2)}(s)ds = 1 + \int_0^t a\left(1 + as + \frac{a^2}{2}s^2\right)ds = 1 + at + \frac{a^2}{2}t^2 + \frac{a^3}{3!}t^3.
\end{aligned}$$

In fact, we can show that

$$x^{(n)}(t) = 1 + at + \cdots + \frac{a^n}{n!}t^n$$

which converges to the exact solution $x(t) = e^{at}$. The n-th iteration is exactly the n-th Taylor series expansion at the start time $t = 0$. In general, higher order (greater than n) terms may appear in $x^{(n)}(t)$, which may not agree with the Taylor expansion with more than n terms, as shown in the following example.

Example 2.24. Find a power series expansion for solutions to

$$\dot{x} = x - x^2, \quad x_0 = 2$$

correct up to and including cubic terms. Set $x^{(0)}(t) = 2$. Then

$$x^{(1)}(t) = 2 + \int_0^t (2 - 2^2)\mathrm{d}s = 2 - 2t.$$

Continuing

$$x^{(2)}(t) = 2 + \int_0^t \left[(2-2s) - (2-2s)^2\right]\mathrm{d}s = 2 + \int_0^t (-2 + 6s - 4s^2)\mathrm{d}s = 2 - 2t + 3t^2 - \tfrac{4}{3}t^3.$$

Although the cubic term appears in $x^{(2)}(t)$, its coefficient is not that in the Taylor series, and will be correct in the next iteration. That is,

$$\begin{aligned} x^{(3)}(t) &= 2 + \int_0^t (2 - 2s - 3s^2 + \dots) - (2 - 2s - 3s^2 + \dots)^2 \mathrm{d}s \\ &= 2 + \int_0^t \left(-2 + 6s - 13s^2 + 16s^3 - \frac{43}{3}s^4 + 8s^5 - \frac{16}{9}s^6\right)\mathrm{d}s \\ &= 2 - 2t + 3t^2 - \tfrac{13}{3}t^3 + 4t^4 + \cdots, \end{aligned}$$

which is correct to the cubic term. This ODE can be integrated explicitly (a separable ODE) to give

$$x(t) = \frac{2}{2 - e^{-t}} = 2 - 2t - \frac{13}{3}t^3 + \frac{25}{4}t^4 - \frac{541}{60}t^5 + O(t^6).$$

Remark. After some technical work, the existence of solution can be established using the above Picard Iteration scheme $x^{(n+1)}(t) = T[x^{(n)}](t)$ by taking the limit as n goes to infinity. We will focus on qualitative properties in the rest of the course[2], and the interested readers may consult Chapter 3 in Meiss's book *differential dynamical systems*.

[2]You can safely ignore related questions about the existence and uniqueness of ODEs in the past papers.