1. (a) The system can be written as

\[
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix},
\]

with the solution \( \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \). Therefore, the behaviour of the solution is determined by that of \( A^n \). If we write the matrix \( A \) as

\[
A = \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix} = \sqrt{1 + h^2} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \theta = \arctan h \in (0, \pi/2),
\]

then the action of the matrix \( A \) in the map \( \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix} \) is to rotate the vector \( \begin{pmatrix} x_n \\ y_n \end{pmatrix} \) by an angle \( \theta \) in clockwise direction and dilate the length by a factor of \( \sqrt{1 + h^2} \). The solution can also be written as

\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (1 + h^2)^{n/2} \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

Then \( (x_n, y_n) \) goes to infinity because

\[
x_n^2 + y_n^2 = (1 + h^2)^n \left[ (x_0 \cos n\theta + y_0 \sin n\theta)^2 + (-x_0 \sin n\theta + y_0 \cos n\theta)^2 \right] = (1 + h^2)^n (x_0^2 + y_0^2).
\]

Alternatively, the matrix power \( A^n \) can be calculated by diagonalisation with the eigenvectors: if \( e_1 \) and \( e_2 \) are the two eigenvectors associated with the two eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then \( A = S \text{diag}(\lambda_1, \lambda_2) S^{-1} \) with \( S = [e_1 \ e_2] \) and \( A^n = S \text{diag}(\lambda_1^n, \lambda_2^n) S^{-1} \). The two eigenvalues of \( A \) are governed by \( \det(\lambda I - A) = (\lambda - 1)^2 + h^2 = 0 \) or \( \lambda_{\pm} = 1 \pm ih \), with two eigenvectors

\[
e_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad e_- = \begin{pmatrix} 1 \\ -i \end{pmatrix}.
\]

Then the matrix \( A \) can be diagonalised as

\[
A = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \lambda_+ & 1 \\ 1 & \lambda_- \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1},
\]

and

\[
A^n = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \lambda_+^n & \lambda_-^n \\ \lambda_-^n & \lambda_+^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \lambda_+^n & i(\lambda_-^n - \lambda_+^n) \\ \lambda_-^n & \lambda_+^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}
\]

\[
= \frac{1}{2} \begin{pmatrix} \lambda_+^n + \lambda_-^n & i(\lambda_-^n - \lambda_+^n) \\ i(\lambda_-^n - \lambda_+^n) & \lambda_+^n + \lambda_-^n \end{pmatrix}.
\]
Since $|\lambda_+| = |\lambda_-| = \sqrt{1 + h^2} > 1$, every entry in the non-singular matrix $A^n$ converges to infinity as $n$ goes to infinity. Therefore, the solution $(x_n, y_n)$ goes to infinity for any non-zero initial condition $(x_0, y_0)$.

(b) The system can be written as

$$
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} - hy_{n+1} \\ y_{n+1} + hx_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}
$$

or equivalently

$$
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = B \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix}^{-1} = \frac{1}{1+h^2} \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix} = \frac{1}{1+h^2} A.
$$

Using the results from (a),

$$
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (1+h^2)^{-n} A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (1+h^2)^{-n/2} \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},
$$

and

$$
x_n^2 + y_n^2 = (1+h^2)^{-n} [(x_0 \cos n\theta + y_0 \sin n\theta)^2 + (-x_0 \sin n\theta + y_0 \cos n\theta)^2] = (1+h^2)^{-n}(x_0^2+y_0^2) \to 0,
$$
as $n$ goes to infinity.

(c) The system can be written as

$$
x_{n+1} = x_n + hy_n, \quad y_{n+1} = y_n - hx_{n+1} = (1-h^2)y_n - hx_n.
$$

If $Q(x_n, y_n) = Ax_n^2 + Bx_ny_n + y_n^2$ is conserved for this system, then

$$
0 = Q(x_{n+1}, y_{n+1}) - Q(x_n, y_n)
$$

$$
= Ax_{n+1}^2 + Bx_{n+1}y_{n+1} + y_{n+1}^2 - (Ax_n^2 + Bx_ny_n + y_n^2)
$$

$$
= A(x_n + hy_n)^2 + B(x_n + hy_n)((1 - h^2)y_n - hx_n) + ((1 - h^2)y_n - hx_n)^2 - (Ax_n^2 + Bx_ny_n + y_n^2)
$$

$$
= (h^2 - Bh)x^2 + (2Ah - 2h - 2bh^2 + 2h^3)xy + (Bh + Ah^2 - Bh^3 - 2h^2 + h^4)y^2,
$$

which is valid if and only if the coefficients vanishes identically, i.e.,

$$
h^2 - Bh = 0, \quad 2Ah - 2h - 2Bh^2 + 2h^3 = 0, \quad Bh + Ah^2 - Bh^3 - 2h^2 + h^4 = 0.
$$

The first equation implies that $B = h$, and the rest two equations are reduced to

$$
2Ah - 2h = 0, \quad Ah^2 - h^2 = 0,
$$

implying that $A = 1$. Therefore, the conserved quantity is $Q(x_n, y_n) = x_n^2 + y_n^2 + hx_ny_n$.

2. Let $V(x, y, z) = Ax^2 + By^2 + Cz^2$, then

$$
\dot{V} = 2Ax\dot{x} + 2By\dot{y} + 2Cz\dot{z} = (2A - 4B - 8C)xyz.
$$
Since the sign of $xyz$ could be arbitrary, we can only choose $2A - 4B - 8C = 0$, such that the set $\{(x, y, z) \mid V(x, y, z) = r_0 \}$ is invariant for any constant $r_0 > 0$. As a result, the set $U = \{(x, y, z) \mid V(x, y, z) < r_0 \}$ is also invariant. In other words, if $(x_0, y_0, z_0)$ is in $U$, so is the solution $(x(t), y(t), z(t))$. Therefore, we only have to choose the constants $A, B, C$ and $r_0$, such that $U$ is a subset of the open unit ball $S$, i.e.

$$U = \{(x, y, z) \mid Ax^2 + By^2 + Cz^2 < r_0 \} \subset S = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1 \}.$$  

This condition requires that (assuming $r_0 > 0$)

$$A \geq r_0, \quad B \geq r_0, \quad C \geq r_0,$$

together with the condition $2A - 4B - 8C = 0$ for $U$ to be invariant.

There are many different choices, including the simplest one $A = 6r_0$, $B = r_0$ and $C = r_0$, giving the set

$$U = \{(x, y, z) \mid r_0(6x^2 + y^2 + z^2) < r_0 \} = \{(x, y, z) \mid 6x^2 + y^2 + z^2 < 1 \}.$$

3. a) From the definition,

$$W(\tau) = \frac{d}{d\tau} U(\tau) = r \frac{d}{dr} \left( u(r)r^{q/(q-2)} \right) = u'(r)r^{q/(q-2)} + \frac{2}{q-2} u(r)r^{q/(q-2)} = u'(r)r^{q/(q-2)} + \frac{2}{q-2} U(\tau),$$

which implies that $u'(r)r^{q/(q-2)} = W(\tau) - \frac{2}{q-2} U(\tau)$. Taking the derivative of $W$ w.r.t $\tau$, we get

$$\frac{d}{d\tau} W(\tau) = r \frac{d}{dr} \left( u'(r)r^{q/(q-2)} \right) + \frac{2}{q-2} \frac{d}{d\tau} U(\tau)$$

$$= u''(r)r^{(2q-2)/(q-2)} + \frac{q}{q-2} u'(r)r^{q/(q-2)} + \frac{2}{q-2} W(\tau).$$

From the governing ODE for $u(r)$, we get

$$u''(r)r^{(2q-2)/(q-2)} = -(N-1)u'(r)r^{q/(q-2)} - u(r)q^{-1}r^{(2q-2)/(q-2)} = (1-N)u'(r)r^{q/(q-2)} - U(\tau)^{q-1},$$

and hence

$$\frac{d}{d\tau} W(\tau) = \left( \frac{q}{q-2} + 1 - N \right) u'(r)r^{q/(q-2)} - U(\tau)^{q-1} + \frac{2}{q-2} W(\tau)$$

$$= \left( \frac{q}{q-2} + 1 - N \right) \left( W(\tau) - \frac{2}{q-2} U(\tau) \right) - U(\tau)^{q-1} + \frac{2}{q-2} W(\tau)$$

$$= \left( \frac{2q}{q-2} - N \right) W(\tau) - \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) U(\tau) - U(\tau)^{q-1}.$$ 

Therefore, the system of ODEs is

$$\dot{U} = W, \quad \dot{W} = \left( \frac{2q}{q-2} - N \right) W - \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) U - U^{q-1}.$$
b) The linearised system near the origin is (the term $U^{q-1}$ is higher order, because $q > 2$)

\[
\frac{d}{d\tau} \begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) & 2q - N \end{pmatrix} \begin{pmatrix} U \\ W \end{pmatrix}
\]
and the eigenvalues are the roots of the quadratic function

\[
\det \begin{pmatrix} \lambda & -1 \\ \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) & 2q - N \end{pmatrix} = \lambda^2 + \left( -\frac{2q}{q-2} + N \right) \lambda + \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right).
\]

By inspection, the two eigenvalues are $\lambda_1 = 2/(q-2)$, $\lambda_2 = q/(q-2) + 1 - N$.

c) Using the relation $\tau = \log r$, the limit can be written as

\[
k = \lim_{\tau \to -\infty} \frac{W(\tau)}{U(\tau)} = \lim_{r \to 0} \frac{u'(r) r^{q/(q-2)} + \frac{2}{q-2} u(r) r^{2/(q-2)}}{u(r) r^{2/(q-2)}} = \frac{2}{q-2} + \lim_{r \to 0} \frac{ru'(r)}{u(r)} = 0.
\]

Then if we multiply the coefficient matrix in part b) with the vector $\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ 2/(q-2) \end{pmatrix}$,

\[
\begin{pmatrix} \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) & 1 \\ -\frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) & \frac{2q}{q-2} - N \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{q-2} \end{pmatrix} = \begin{pmatrix} 2/(q-2) - \frac{2}{q-2} \left( \frac{q}{q-2} + 1 - N \right) + \frac{2}{q-2} \left( \frac{2q}{q-2} - N \right) \\ \frac{2}{q-2} \left( \frac{1}{q-2} \right) \end{pmatrix}
\]

implying that this is an eigenvector associated with the eigenvalue $2/(q-2)$.

d) In this special case $N = 3$ and $q = 3$, the system becomes

\[
\dot{U} = W, \quad \dot{W} = 3W - 2U - U^2.
\]

Since the unstable manifold passes through the origin, and is tangent to the eigenvector $\begin{pmatrix} 1 \\ 2/(q-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, it can be represented as

\[
W = 2U + aU^2 + \cdots
\]

The invariant condition implies that

\[
0 = \frac{d}{d\tau} (2U + aU^2 + \cdots - W) \big|_{W=2U+aU^2+\cdots} = \frac{d}{d\tau} (2U + aU) W - (3W - 2U - U^2) \big|_{W=2U+aU^2+\cdots} = \frac{d}{d\tau} (2U + aU^2 + \cdots) = (3a + 1)U^2 + \cdots,
\]

or $a = -1/3$.

Remark: The unstable manifold can also be represented as

\[
0 = \phi(U, W) = W - 2U + aU^2 + bUW + cW^2 + \cdots
\]
with infinitely many choices of constants \(a, b\) and \(c\). Besides the one above (with \(b = c = 0\)), another choice is to consider

\[
\frac{d}{d\tau} \phi(U, W) = \left( (-2 + 2aU + bW + \cdots) \frac{\partial}{\partial U} + (1 + bU + 2cW + \cdots) \frac{\partial}{\partial W} \right)
\]

\[
= (-2 + 2aU + bW + \cdots) W + (1 + bU + 2cW + \cdots)(3W - 2U - U^2)
\]

\[
= W - 2U + (-1 - 2b)U^2 + (2a + 3b - 4c)UW + (b + 6c)W^2 + O(U^3, W^3).
\]

Since we need \(\frac{d}{d\tau} \phi(U, W) = 0\) on the set \(\phi(U, W) = 0\), one choice to find the coefficients \(a, b, c\) is to use the condition

\[
\frac{d}{d\tau} \phi(U, W) = \phi(U, W) = 0
\]

implying that

\[
a = -1 - 2b, \quad b = 2a + 3b - 4c, \quad c = b + 6c
\]

leading to \(a = 7/3, b = -5/3\) and \(c = 1/3\).

e) When \(q = 2N/(N - 2)\), the equation for \(\dot{W}\) becomes

\[
\dot{W} = -\frac{N - 2}{2} \left( 1 - \frac{N}{2} \right) U - \frac{U^{(N+2)/(N-2)}}{W}
\]

The unstable manifold is a special orbit, governed by the ODE

\[
\frac{dW}{dU} = -\frac{N - 2}{2} \left( 1 - \frac{N}{2} \right) U - \frac{U^{(N+2)/(N-2)}}{W}
\]

which is separable. Integrating both sides of

\[
\left( -\frac{N - 2}{2} \left( 1 - \frac{N}{2} \right) U - \frac{U^{(N+2)/(N-2)}}{W} \right) dU = WdW
\]

we get the equation for the unstable manifold

\[
\frac{1}{2} W^2 + \frac{N - 2}{4} \left( 1 - \frac{N}{2} \right) U^2 + \frac{N - 2}{2N} U^{2N/(N-2)} = C.
\]

Since this orbit passes through the origin, the constant \(C\) vanishes, and the equation is

\[
0 = \frac{1}{2} W^2 + \frac{N - 2}{4} \left( 1 - \frac{N}{2} \right) U^2 + \frac{N - 2}{2N} U^{2N/(N-2)} = \frac{1}{2} W^2 - \frac{(N - 2)^2}{8} U^2 + \frac{N - 2}{2N} U^{2N/(N-2)}.
\]

4. (a) The fixed points satisfy \(y(13 - x^2 - y^2) = 0, 12 - x(13 - x^2 - y^2) = 0\). From the second equation \(12 = x(13 - x^2 - y^2) \neq 0\), we get \(13 - x^2 - y^2 \neq 0\). Therefore, \(y = 0\) and \(12 - 13x + x^3 = 0\). Using the formula for cubic roots or just inspection, we get all three roots:

\[
x_1 = -4, \quad x_2 = 1, \quad x_3 = 3.
\]

The Jacobian matrix is

\[
J(x, y) = \begin{pmatrix}
-2xy & 13 - x^2 - 3y^2 \\
-13 + 3x^2 + y^2 & 2xy
\end{pmatrix}.
\]
At the three fixed points \((x_l, 0)\), we have

\[
J(x_k, 0) = \begin{pmatrix}
0 & 13 - x_k^2 \\
-13 + 3x_k^2 & 0
\end{pmatrix}.
\]

The fixed point \((x_k, 0)\) is a saddle, if and only if the two eigenvalues have opposite signs, or the determinant is negative. That is, the fixed point \((x_k, 0)\) is a saddle, if and only if

\[
\det J(x_k, 0) = (13 - x_k^2)(13 - 3x_k^2) < 0,
\]

which is only satisfied for \(x_3 = 3\).

(b) Near the fixed point \((3, 0)\), the Jacobian matrix is \(J(3, 0) = \begin{pmatrix} 0 & 4 \\ 14 & 0 \end{pmatrix}\) with two eigenvalues \(\lambda_\pm = \pm 2\sqrt{14}\) and the associated eigenvectors

\[
e_+ = \left(1, \lambda_+/4\right), \quad e_- = \left(1, \lambda_-/4\right).
\]

Here, by normalising the first component in the eigenvector \(e_\pm\) to be unit, the second component is exactly the slope.

The stable manifold \(E^s\) for the linearised system is a straight line passes through the fixed point \((3, 0)\) and parallel to \(e_-\) (associated with the negative eigenvalue \(\lambda_-\)). To simplify the presentation, the shift of coordinates \(X = x - 3\), \(Y = y\) is used, and the original equation becomes

\[
\dot{X} = Y(13 - (X + 3)^2 - Y^2) = 4Y - 6XY - Y(X^2 + Y^2),
\]

and

\[
\dot{Y} = 12 - (X + 3)(13 - (X + 3)^2 - Y^2) = 14X + 9X^2 + 3Y^2 + X(X^2 + Y^2).
\]

Since the slope of the line parallel to \(e_-\) is exactly \(\lambda_-/4\),

\[
E^s = \left\{ (X, Y) \mid Y = \frac{\lambda_-}{4}X \right\} = \left\{ (x, y) \mid y = \frac{\lambda_-}{4}(x - 3) \right\}.
\]

Therefore, the stable manifold \(W^s\) for the full system can be represented as

\[
W^s = \left\{ (X, Y) \mid Y = h(X) = \frac{\lambda_-}{4}X + c_2X^2 + \cdots \right\}.
\]

The coefficient \(c_2\) can be obtained using the fact that \(W^s\) is invariant, that is

\[
\frac{d}{dt}(Y - h(X))\bigg|_{Y=h(X)} = (\dot{Y} - h'(X)\dot{X})\bigg|_{Y=h(X)}.
\]

On one hand,

\[
\dot{Y}\big|_{Y=h(X)} = 14X + 9X^2 + 3Y^2 + X(X^2 + Y^2)\big|_{Y=h(X)}
\]

\[
= 14X + 9X^2 + 3\left(\frac{\lambda_-}{4}X + \cdots\right)^2 + X\left(X^2 + \left(\frac{\lambda_-}{4}X + \cdots\right)^2\right)
\]

\[
= 14X + \left(9 + \frac{3\lambda_-^2}{16}\right)X^2 + O(X^3).
\]
On the other hand,

\[ h'(X)X|_{Y=h(X)} = \left(\frac{\lambda_-}{4} + 2c_2X + \cdots\right) \left[4 \left(\frac{\lambda_-}{4}X + c_2X^2 + \cdots\right) - 6X \left(\frac{\lambda_-}{4}X + c_2X^2 + \cdots\right) + \cdots\right] \]

\[ = \left(\frac{\lambda_-}{4} + 2c_2X\right) \left[\lambda_- X + \left(4c_2 - \frac{3\lambda_-}{2}\right)X^2 + \cdots\right] \]

\[ = \frac{\lambda_-^2}{4}X + \left(3c_2\lambda_- - \frac{3\lambda_-^2}{8}\right)X^2 + O(X^3).\]

Then the matching conditions for the coefficients of \(X\) and \(X^2\) become

\[ O(X) : \quad 14 = \frac{\lambda_-^2}{4}, \]

\[ O(X^2) : \quad 9 + \frac{3\lambda_-^2}{16} = 3c_2\lambda_- - \frac{3\lambda_-^2}{8}. \]

The first equation is satisfied identically because \(\lambda_- = -2\sqrt{14}\). The value of \(c_2\) is determined from the second equation as

\[ c_2 = \frac{1}{3\lambda_-} \left(9 + \frac{3\lambda_-^2}{16} + \frac{3\lambda_-^2}{8}\right) = \frac{81}{6\lambda_-} = -\frac{27\sqrt{14}}{56}. \]

Therefore, the stable manifold (correct up to and including the quadratic terms) is

\[ W^s = \left\{ (x, y) \mid y = -\frac{\sqrt{14}}{2}(x - 3) - \frac{27\sqrt{14}}{56}(x - 3)^2 + \cdots \right\}. \]