1. [20 marks] The solution of the continuous system

\[ \dot{x} = y, \quad \dot{y} = -x, \quad x(0) = x_0, \quad y(0) = y_0 \]

can be expressed as a linear combination of \( \sin t \) and \( \cos t \) and hence the solution is bounded. In fact, the solution stay on the circle \( x^2 + y^2 = x_0^2 + y_0^2 \). But a specific numerical approximation, the solution \((x_n, y_n)\) of \((x(t), y(t))\) at \( t = nh \) for some time step \( h > 0 \) may not always be bounded as shown in the following questions.

(a) [6 marks] If the straightforward first order approximation \( x(t_n + h) \approx x(t_n) + hx'(t_n), y(t_n + h) \approx y(t_n) + hy'(t_n) \) is used (the so-called forward Euler Method), we get the system

\[ x_{n+1} = x_n + hy_n, \quad y_{n+1} = y_n - hx_n. \]

Show that the solution \((x_n, y_n)\) goes to infinity for fixed initial condition \((x_0, y_0) \neq (0, 0)\) and fixed time step \( h > 0 \).

(b) [6 marks] If the first order approximation \( x(t_n + h) \approx x(t_n) + hx'(t_n + h), y(t_n + h) \approx y(t_n) + hy'(t_n + h) \) is used (the so-called implicit Euler method), we get the system

\[ x_{n+1} = x_n + hy_{n+1}, \quad y_{n+1} = y_n - hx_{n+1}. \]

Show that the solution \((x_n, y_n)\) goes to the origin, as \( n \) goes to infinity.

(c) [8 marks] If we ”mix” the previous two methods to construct semi-implicit Euler method

\[ x_{n+1} = x_n + hy_n, \quad y_{n+1} = y_n - hx_{n+1}. \]

Show that there is a conserved quadratic function \( Q(x_n, y_n) = Ax_n^2 + Bx_ny_n + y_n^2 \) for some constants \( A \) and \( B \). Find the constants \( A \) and \( B \).

2. [15 marks] Let \((x(t), y(t), z(t))\) be the solution of the system

\[ \dot{x} = yz, \quad \dot{y} = -2xz, \quad \dot{z} = -4xy \]

with initial condition \((x_0, y_0, z_0)\) and \( S = \{(x, y, z)|x^2 + y^2 + z^2 < 1\} \) be the unit open ball in \( \mathbb{R}^3 \). Show that there exists an open set \( U \subset S \), such that if \((x_0, y_0, z_0) \in U \), then \((x(t), y(t), z(t)) \in S \).
for all $t > 0$. Hint: construct an appropriate quadratic Lyapunov function $V(x, y, z)$, and define the set as $U = \{(x, y, z) \mid V(x, y, z) < r_0\}$ for some constant $r_0 > 0$ (which you have to find).

3. [40 marks] We continue investigating how radial solutions of the semi-linear elliptic partial differential equations $-\Delta u = u^{q-1}$ on the unit ball $|x| < 1$ depend on the dimension $N$ and the exponent $q$. Recall that the radial solution $u(r)$ satisfies the second order ODE

$$\frac{d^2}{dr^2} u(r) + \frac{N-1}{r} \frac{du}{dr}(r) + u(r)^{q-1} = 0, \quad N > 2, \ q > 2, \quad (*)$$

for $r \in (0, 1)$ and the desired solution $u(r)$ is smooth, non-negative on the interval $(0, 1)$, satisfying the conditions $u(0) > 0$, $u'(0) = 0$ and $u(1) = 0$.

![Graph of u(r) and W](image)

Figure 1: The solution $u(r)$ governed by the second order non-autonomous ODE is transformed into a trajectory governed by a first order autonomous system for $U$ and $W$.

a) [8 marks] In the first coursework, the change of variables $\tau = \log r, U(\tau) = u(r) r^{\frac{q-2}{2}}, V(\tau) = u'(r) r^{\frac{q-2}{2}}$ leads to the autonomous system

$$\dot{U} = \frac{2}{q-2} U + V, \quad \dot{V} = -U^{q-1} + \left(\frac{q}{q-2} + 1 - N\right) V.$$ 

If we choose the following slightly different change of variables

$$\tau = \log r, \quad U(\tau) = u(r) r^{\frac{q-2}{2}}, \quad W(\tau) = \frac{d}{d\tau} U(\tau),$$

find the autonomous system governing the variables $U$ and $W$.

b) [8 marks] Find the two eigenvalues of the Jacobian matrix near the origin. The resulting quadratic characteristic polynomial can be factorised, and you may just get them by inspection, instead of using the more complicated formula for the roots of a quadratic equation (the discriminant turns out to be a “perfect square”).
c) [8 marks] Let \( k \) be the limit \( \lim_{\tau \to -\infty} \frac{W(\tau)}{U(\tau)} \). Find this limit, by writing it in terms of \( r \) where \( U \) and \( W \) are expressed using the desired solution \( u \) (which is assumed to be smooth). Show that the vector \( \begin{pmatrix} 1 \\ k \end{pmatrix} \) is an eigenvector of the coefficient matrix for the linearised system in b), and find the corresponding eigenvalue.

d) [8 marks] The desired radial solution \( u \) corresponds an unstable manifold near the origin that intersects the \( W \)-axis (the condition \( u(1) = 0 \) becomes \( U(0) = 0 \)). Find a local approximation of this unstable manifold for the special case of \( N = 3 \) and \( q = 3 \), correct up to and including quadratic terms.

e) [8 marks] There is a global bifurcation point at \( q = q^* = 2N/(N-2) \) (the so called critical Sobolev exponent), such that there is no solution for \( q \geq q^* \). When \( q = q^* = 2N/(N-2) \), the unstable manifold returns to the origin (a homoclinic orbit), instead of intersecting the negative \( W \)-axis. Find the equation for this special orbit. Hint: simplify the equation for \( \dot{W} \) using \( q = 2N/(N-2) \) before doing any calculation.

4. [25 marks] Consider the following system of differential equations

\[
\dot{x} = y(13 - x^2 - y^2), \quad \dot{y} = 12 - x(13 - x^2 - y^2).
\]

(a) [10 marks] Find the only fixed point whose linear type is a saddle. You can find the formula of roots to cubic polynomials online (Cardano’s formula), but it is much easier to get the roots in this problem just by inspection (The coordinates of all fixed points are integers with absolute values not larger than 4).

(b) [15 marks] Find the stable manifold around the only saddle you found in (a), correct up to and including the quadratic terms. Write your final result in terms of \( x \) and \( y \).

You may use books and the internet as sources provided they are acknowledged and you may discuss the problems with other students. But the work handed in must be your own; see the University Guidelines on Plagiarism for more details: