MATH36032 Problem Solving by Computer

Synchronization
Opened in June 2000, but then was \textbf{closed} for almost two years ...
Here is what happened on the opening day ...

https://www.youtube.com/watch?v=eAXVa__XWZ8
The Mathematical Model

the bridge: \[ M \frac{d^2 X}{dt^2} + B \frac{dX}{dt} + KX = G \sum_{j=1}^{N} \sin \Theta_j, \]

the gait: \[ \frac{d\Theta_j}{dt} = \Omega_j + CA \sin(\psi - \Theta_j + \alpha), j = 1, 2, \cdots, N. \]

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A closer look at this (useful) model: the bridge

“All models are wrong but some are useful.”

— Statistician George E. P. Box (1919 – 2013)

The bridge is a weakly damped and driven harmonic oscillator

\[ M \frac{d^2 X}{dt^2} = -B \frac{dX}{dt} - KX + G \sum_{j=1}^{N} \sin \Theta_j. \]

- \( X \): the lateral displacement
- \( M, B, K \): the effective mass, damping, and stiffness
- \( G \): maximum force of each pedestrian
- \( \Theta_j \): the gait (phase) of each pedestrian
A closer look at this (useful) model: the pedestrian

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The equation for individual pedestrian’s gait:

\[
\frac{d\Theta_j}{dt} = \Omega_j + CA \sin(\Psi - \Theta_j + \alpha).
\]

▶ Θ_j, Ω_j: the gait (phase) and the pace of each pedestrian

▶ C: pedestrian’s sensitivity to bridge vibrations

▶ A(t), Ψ(t): the amplitude and the phase of the vibration of the bridge: \( X = A \sin \Psi, \frac{dX}{dt} = \sqrt{K/MA} \cos \Psi. \)

▶ α: phase lag parameter
The whole system and the parameters

\[ M \frac{d^2 X}{dt^2} + B \frac{dX}{dt} + KX = \sum_{j=1}^{N} G \sin \Theta_j, \]
\[ \frac{d\Theta_j}{dt} = \Omega_j + CA \sin (\Psi - \Theta_j + \alpha), j = 1, 2, \cdots, N. \]

Find \([X \; \dot{X} \; \Theta_1 \; \Theta_2 \; \cdots \; \Theta_N]\) with the following parameters:

- \(M = 1.13 \times 10^5 \text{kg}, B = 1.1 \times 10^4 \text{kg/s}, K = 4.73 \times 10^6\)
- \(G = 30 \text{kg} \cdot \text{m/s}^2, C = 16 \text{m}^{-1} \text{s}^{-1}, \alpha = \pi/2\)
- \(A = \sqrt{X^2 + M\dot{X}^2}/K, \tan \Psi = \frac{X}{\dot{X}} \sqrt{\frac{K}{M}}, \Omega_0 = \sqrt{K/M} \approx 6.47 \text{rad/s}\)
- \(\Omega_j \in \mathcal{N}(\Omega_0, 0.63 \text{rad/s})\) (normal distribution), \(\Theta_j(0)\) random

Two quantities of interests: the **wobble amplitude** \(A\) and the **synchronization parameter** \(R\):

\[ A = \sqrt{X^2 + \frac{M}{K} \left(\frac{dX}{dt}\right)^2}, \quad R = \left| \frac{1}{N} \sum_{j=1}^{N} \exp(i\Theta_j) \right|. \]
Kuramoto model

A related **kuramoto model** (for the flashing of fireflies):

\[
\frac{d\Theta_j}{dt} = \Omega_j + \frac{\beta}{N} \sum_{k=1}^{N} \sin(\Theta_k - \Theta_j), \quad j = 1, 2, \cdots, N.
\]

Synchronization happens when $\beta$ is large enough!
ODE function for \( \frac{d\Theta_j}{dt} = \Omega_j + \frac{\beta}{N} \sum_{k=1}^{N} \sin(\Theta_k - \Theta_j) \)

You can define the right hand side of the ode using for loops, but also more elegantly using kron (Kronecker product) or repmat, by taking the (row) sum of the following:

\[
\begin{align*}
\sin & \left( \begin{bmatrix}
\Theta_1 & \Theta_2 & \cdots & \Theta_N \\
\Theta_1 & \Theta_2 & \cdots & \Theta_N \\
\vdots & \vdots & \ddots & \vdots \\
\Theta_1 & \Theta_2 & \cdots & \Theta_N \\
\end{bmatrix} - \\
\begin{bmatrix}
\Theta_1 & \Theta_1 & \cdots & \Theta_1 \\
\Theta_2 & \Theta_2 & \cdots & \Theta_2 \\
\vdots & \vdots & \ddots & \vdots \\
\Theta_N & \Theta_N & \cdots & \Theta_N \\
\end{bmatrix} \\
\end{align*}
\]

```matlab
function dzdt = kuraNode(t,z,omega,beta)
% z = [theta1; theta2; .. thetaN] (a column vector)
dzdt = omega + ... 
    beta*sum(sin(kron(ones(length(z),1),z') ...
    -kron(z,ones(1,length(z)))),2)/length(z);
```