

Review Problems for the second Midterm ¹

(No solution is given. Please ask the instructor or the TA in case you have a question.)

1. Consider the objective function

$$f(x) = x_1^2 + x_1^2 x_3^2 + 2x_1 x_2 + x_2^4 + 8x_2$$

subject to the constraint

$$c(x) = 2x_1 + 5x_2 + x_3 = 3.$$

- (a) Determine which of the following points are stationary points: $(0, 0, 2)$; $(0, 0, 3)$; $(1, 0, 1)$. (x^* is called a stationary point, if it satisfies the first order constraint, i.e., in the feasible set and $\nabla f(x^*) = \lambda \nabla c(x^*)$ for some scalar λ).
 - (b) Determine whether each of the stationary point is a local minimizer, a local maximizer or a saddle point.
2. Determine the minimizers/maximizers (from the second order conditions) of the following functions subject to the given constraints.
 - (a) $f(x) = x_1 x_2^3$ subject to $2x_1 + 3x_2 = 4$
 - (b) $f(x) = x_1^3 + x_2^3$ subject to $2x_1 + x_2 = 1$
 - (c) $f(x) = 2x_1 - 3x_2$ subject to $x_1^2 + x_2^2 = 25$
 - (d) $f(x) = x_2$ subject to $x_1^3 + x_2^3 - 3x_1 x_2 = 0$

3. Let a_1, a_2, a_3 be positive constants, find the **maximal value** of $x_1 x_2 x_3$ subject to the constraints

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_2}{a_2} = 1, \quad x_1, x_2, x_3 \geq 0.$$

You can assume the constraints $x_1, x_2, x_3 \geq 0$ are *inactive* in your calculation (no Lagrange Multipliers for them or assume the corresponding Lagrange Multipliers are all zero). Show that the second order sufficient condition is satisfied at the minimizer you find.

4. Let $a_1, a_2, a_3, c_1, c_2, c_3$ be positive constants, find

$$\min f(x) = \frac{c_1}{x_1} + \frac{c_2}{x_2} + \frac{c_3}{x_3}$$

subject to the constraints

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 1, \quad x_1, x_2, x_3 \geq 0.$$

You can assume the constraints $x_1, x_2, x_3 \geq 0$ are *inactive* in your calculation (no Lagrange Multipliers for them or assume the corresponding Lagrange Multipliers are all zero). Show that the second order sufficient condition is satisfied at the minimizer you find.

¹Some of the problems may seem difficult, but the those in the midterm (and final) should be simpler, at least on the computational side.

5. Consider the following problem

$$\begin{array}{ll} \min & f(x) = \frac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} & 2x_1 + x_2 \geq 2 \quad (c_1) \\ & -x_1 + x_2 \geq -1 \quad (c_2) \\ & x_1 \geq 0. \quad (c_3) \end{array}$$

Given that c_2 and c_3 are inactive at the global minimizer x^* , is c_1 active at x^* ? Find x^* and show the second order sufficient condition is satisfied.

6. Consider the problem

$$\begin{array}{ll} \min & f(x) = \|x\|_1 = |x_1| + |x_2| \\ \text{subject to} & 1 - (x_1 - 1)^2 - (x_2 - 1)^2 \geq 0. \end{array}$$

- (a) Plot the feasible region and the level set (contour) of the object function.
- (b) Find the global minimizer use graphic method.
- (c) Show that the minimizer satisfies the first and second sufficient optimal conditions. (Hint: Though $|x_1|$ is not differentiable, but it is away from $x_1 = 0$. You can get rid of the absolute value signs by assuming (x_1, x_2) varies near the minimizer x^* and you can get all the derivatives).

7. Consider the problem

$$\begin{array}{ll} \min & f(x) = e^x \\ \text{subject to} & x \geq 1. \end{array}$$

- (a) Write this as an equivalent min-max problem of the Lagrange function (be careful about the constraint on both variables x and λ).
- (b) Find the dual problem by changing it to a max-min problem
- (c) Find the optimizer x^* and λ^*

8. Find the dual problem of the linear program

$$\begin{array}{ll} \min & c^t x \\ \text{subject to} & l \leq x \leq u. \end{array}$$

where c, l, u are constant vectors in \mathbb{R}^n . Find the solution of the primal and the dual problem.

9. The dual problem is not unique, depending on how many dual variables (Lagrange Multipliers) we introduce for the constraints. We may not introduce Lagrange Multipliers for some simple constraints like $x_i \geq 0$.

For the primal problem

$$\begin{aligned} \min \quad & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} \quad & x_1 + x_2 + 2 \geq 0 \\ & x_1 \geq 0. \end{aligned}$$

Depending on whether we introduce a Lagrange Multiplier for the constraint $x_1 \geq 0$ or not, the original problem is equivalent to

$$\min_x \max_{\lambda_1, \lambda_2 \geq 0} L(x, \lambda) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \lambda_1(x_1 + x_2 + 2) - \lambda_2 x_1$$

and

$$\min_{x_1 \geq 0} \max_{\lambda_1} \tilde{L}(x, \lambda) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \lambda_1(x_1 + x_2 + 2)$$

The dual problem for the first formulation is

$$\max_{\lambda_1, \lambda_2 \geq 0} q(\lambda_1, \lambda_2)$$

where q is defined as

$$q(\lambda_1, \lambda_2) = \min_x \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \lambda_1(x_1 + x_2 + 2) - \lambda_2 x_1.$$

The dual problem for the second formulation is

$$\max_{\lambda_1} \tilde{q}(\lambda_1)$$

where

$$\tilde{q}(\lambda_1) = \min_{x_1 \geq 0} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \lambda_1(x_1 + x_2 + 2).$$

Find q and \tilde{q} , solve both dual problems. (In the second formulation, \tilde{q} has only one variable but the optimization w.r.t x is more difficult because of the constraint $x_1 \geq 0$ in the definition of \tilde{q} . In other words, \tilde{q} is more complicated.)

Past Exams (midterms and finals for years) of Math 309 ²

1. Find the maximum and minimum values of the function $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$. Show that the the minimizer you find satisfies all the first order necessary and second order sufficient conditions.

²Courtesy of Professor Adam Oberman (the instructor). He focused on **Convex Optimization** from the very beginning, which will be covered later in this class. Only the relevant questions are given here.

2. (a) State the Arithmetic-Geometric Mean Inequality for weights w_1, \dots, w_n and values x_1, \dots, x_n . When does equality hold?
- (b) Use the Arithmetic-Geometric Mean Inequality to solve the following Geometric Program. Find the optimal values x^*, y^*, z^* and the value of the objective function ³.

$$\min_{x,y,z>0} x + y^2 + z^4 \quad \text{subject to} \quad xy^2z^4 = 8.$$

- (c) Now solve the problem with modified constraints. You need not find the optimal values ⁴.

$$\min_{x,y,z>0} x + y^2 + z^4 \quad \text{subject to} \quad xyz = 1.$$

3. *Projections, Null and Range Spaces*

- (a) Decompose the vector $x = (2, 2, 1)$ into $x = p + q$, where p is in the direction $(1, 2, 2)$ and q is orthogonal to p .
- (b) Compute a basis matrix for the null space of A where

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \end{pmatrix}$$

- (c) Compute a basis matrix for the null space of A where

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

4. *Optimization with inequality constraints* For each of the following functions and constraints, draw a sketch of the problem, find the maximum and minimum values, and show all the optimality conditions are satisfied for minimizers.

- (a) $f(x, y) = (x - 3)^2 + (y - 5)^2$ subject to $x + y \leq 7$ and $x, y \geq 0$.
- (b) $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

³Taking the constraints $x, y, z \geq 0$ to be inactive. No need to check the optimality conditions.

⁴Taking the constraints $x, y, z \geq 0$ to be inactive. No need to check the optimality conditions.