Review Problems for the first Midterm [1] (No solution is given. Please ask the instructor or the TA in case you have a question.)

Background material about Calculus and Linear Algebra [2]

1. Find the first and second order derivatives (gradient and Hessian matrix) of the following functions:
$$f(x) = \log(1 + x^2), \quad f(x) = \cos(e^x), \quad f(x, y) = \sqrt{x^2 + y^2}.$$

2. Let $A \in \mathbb{R}^{m\times k}, B \in \mathbb{R}^{k\times n}$, show that $(AB)^t = B^t A^t$.

3. Fin the eigenvalues, the associated eigenvectors, the trace and the determinant of the following matrices.
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. If the eigenvalues of a square matrix $A \in \mathbb{R}^{n\times n}$ is $\lambda_1, \lambda_2, \cdots, \lambda_n$, then
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \qquad \det(A) = \lambda_1\lambda_2 \cdots \lambda_n.$$
Verify these statements for the matrices in previous problem.

5. Find the $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$ of the vector x, where
$$x^t = [1, 2, 3] \ \in \ \mathbb{R}^3.$$

6. Show that if
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$
then the inverse of A is given by
$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$
This gives a very clean formula for the inverse of a 2×2 matrix, and in higher dimensions, the formula is much more complicated.

7. Determine whether the following matrix is positive definite, negative definite, non-negative definite (or positive semi-definite), non-positive definite (or negative semi-definite) or neither?
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

[1] A few problems are from the textbook. I'll put it as **suggested problems** on the slides later in this class.

[2] These problems are not standalone, and are actually embedded in the later calculations. Skip this part if you did reasonably well in these prerequisite classes.

Practice problems for the first midterm

1. Find the largest connected interval on which $f(x) = \sqrt{1+x^2} - x$.

2. Show that if f and g are convex on Ω, so is $h(x) = f(x) + g(x)$.

3. Show that both $f(x) = \max(|x-1|, x^2)$ and $g(x) = |x-1| + x^2$ are convex.

4. Find the minimizer of the following problem by graphic method.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x_1, x_2) = x_1 \\ \text{subject to} \quad & x_1^2 \leq x_2, \\ & x_1^2 + x_2^2 \leq 2. \end{aligned}$$

5. Find the maximizer of the following problem by graphic method.

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & f(x_1, x_2) = x_1 + x_2 \\ \text{subject to} \quad & x_1 x_2 \geq 0, \\ & -2 \leq x_1 \leq 1, \\ & -2 \leq x_2 \leq 1. \end{aligned}$$

6. If at the point x^*, f satisfies the condition $\nabla f(x^*) = 0$ and

$$\nabla^2 f(x^*) = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}.$$

Is this a local minimizer or local maximizer or neither? If neither maximizer nor minimizer, can you find two directions p_+ and p_- such that $\phi_+(t)$ is increasing and ϕ_- is decreasing on $t \in [0, \delta)$ for δ small? Here

$$\phi_+(t) = f(x^* + tp_+), \qquad \phi_-(t) = f(x^* + tp_-).$$

7. Find the global minimizer of the function

$$f(x) = x^2 + |x-2|.$$

8. Show that if $\bar{\lambda}$ is a local minimizer of $f(\lambda) = \ln L(\lambda)$ for some positive function $L(\lambda)$, with $f''(\bar{\lambda}) > 0$, then $\bar{\lambda}$ is also a local **minimizer** of L, by showing that $\bar{\lambda}$ satisfies the sufficient condition for L.

9. Show that if $(\bar{\lambda}, \bar{\mu})$ is a local minimizer of $f(\lambda, \mu) = \ln L(\lambda, \mu)$ for some positive function $L(\lambda, \mu)$, with $\nabla^2 f(\bar{\lambda}, \bar{\mu})$ is positive definite, then $(\bar{\lambda}, \bar{\mu})$ is also a local **minimizer** of L, by showing that $(\bar{\lambda}, \bar{\mu})$ satisfies the sufficient condition for L.

10. Let A be positive definite. For the quadratic function

$$f(x) = \frac{1}{2}x^t Ax - b^t x,$$

write down one iteration of the Newton's method for x_1 (in terms of x_0, A and b). How fast the Newton's method converges, in this special case?

11. Let A be positive definite and f be the quadratic function

$$f(x) = \frac{1}{2}x^t Ax - b^t x.$$

If you made a mistake in your code calculating α_k in Steepest Descent method,

$$x_{k+1} = x_k + \alpha_k p_k, \quad p_k = -r_k = b - Ax_k, \quad \alpha_k = \frac{1}{2}\frac{p_k^t p_k}{p_k^t A p_k}.$$

(The actual one $\alpha_k = \dfrac{p_k^t p_k}{p_k^t A p_k}$ should not have the factor 1/2). Does x_k still converges to $x^* = A^{-1}b$?

Past Exams (midterms and finals for years) of Math 309 [3]

1. Let $f(x,y) = \dfrac{1}{12}x^4 + \dfrac{1}{3}y^4$.

   (a) Compute $\nabla f(x,y), \nabla^2 f(x,y)$. Is f convex? Explain your answer.

   (b) Write out the formula for Newton's method for function minimization.

   (c) Plug in the values for f into the Newton iteration and simplify your answer.

   (d) What is the outcome of two Newton iterations, for $(x^0, y^0) = (4, 6)$? Sketch your answer, sketching the level sets of f and mark down each of the iterations.

   (e) What is the result of the kth iteration of Newton's method, starting from the point x_0, y_0?

2. Let $f(x) = \dfrac{1}{4}(x-5)^4 + x$.

   (a) Compute $f'(x), f''(x)$. Is f convex? Explain your answer.

[3] Courtesy of Professor Adam Oberman (the instructor). He focused on **Convex Optimization** from the very beginning, which will be covered later in this class. Only the relevant questions are given here.

(b) Find the minimizer of $f(x)$.

(c) Write out the formula for Newton's method for function minimization.

(d) Compute two Newton iterations, for $x^0 = 4.5$. Are the values approaching the minimum?

3. Let $f(x, y) = 4x^2 + y^2 - 2xy$.

   (a) Compute $\nabla f(x), Hf(x)$. Is f convex? Explain your answer.

   (b) Find the minimizer of $f(x, y)$.

   (c) Write out the formula for the Steepest Descent method for function minimization.

   (d) Compute one Steepest Descent iteration, starting with initial point $(x^0, y^0) = (2, 1)$.

4. Consider the inconsistent system of linear equations

$$\begin{aligned} x + y &= 4 \\ x + 2y &= 6 \\ x + 3y &= 7. \end{aligned}$$

   Find the point (x^*, y^*) that minimizes the error $\|Ax - b\|_2^2$. Calculate $Ax - b$ and the error $\|Ax - b\|_2^2$.

5. Consider the function $f(x, y) = y^2$ on the set S defined by $x^2 + y^2 \leq 4$, and $y \leq |x|$. Sketch the feasible set. Use the sketch to determine all local minimizers and maximizers for the problem, and determine which are also global.

6. *Newton's Method*

   (a) Write out the general formula for Newton's method, and write it out in the special case $f(x) = x^2 - 1$. Simplify the resulting expression.

   (b) Perform two or three (if you have a calculator) iterations of Newton's method, starting with $x_0 = 2$. Which root is it converging to?

7. *Rates of Convergence* For each of the following sequences, write out the first few terms, and find the limit. Then determine the rate of convergence and the rate constant.

   (a) The sequence with general term $x_k = 3^{-k}$.

   (b) The sequence with general term $x_k = 5 + 10^{-2^k}$.