Practice Problems for the final

1. (Projection of a line on a convex set). Let $\Omega = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ be the unit square and $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, g = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the projection of the line $x - tg$ on Ω as a function of $t \in \mathbb{R}$.

2. Using the definition of convexity of a function, show the following facts
   (a) If h and g are convex, then so are $m(x) = \max(f(x), g(x))$ and $h(x) = f(x) + g(x)$
   (b) If f and g are convex and g is non-decreasing, then $h(x) = g(f(x))$ is convex
   (c) If $f(x, y)$ is convex in x then $g(x) = \sup_{y \in C} f(x, y)$ is convex

3. Consider the problem
$$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{subject to } l \leq x \leq u,$$
   where $l, u \in \mathbb{R}^n$ are two constant vectors with $l_i \leq u_i$.
   (a) Show that the first order (KKT) conditions at x^* is equivalent to
$$x^* - P(x^* - \nabla f(x^*), l, u) = 0,$$
   where $P(x, l, u)$ is the projection of x on the rectangular box $\Omega = \{y \mid l_i \leq y_i \leq u_i\}$.
   (b) This also suggest an iterative scheme $x^{n+1} = P(x^n - \nabla f(x^n), l, u)$. For
$$f(x) = x_1^2 - x_1 x_2 - x_2^2, \quad l = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x^0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix},$$
   Calculate x^1 and x^2.

4. *Projections onto planar sets* In each of the problems below, you are given a convex set in the plane, and you are asked to find projection of an arbitrary point x_0 onto the plane. In each case, (i) give sketch of the set, with arrows to show the projection, (ii) give a formula for the projection.
   (a) Let S be the unit square.
   (b) Let C be the intersection of the unit circle, and the half plane whose boundary is the line through $(0, 1)$ and $(1, 0)$ and which contains the origin.

5. *Norm minimization with linear constraints* For each of the following problems, give a sketch of the constraints and the level sets of the objective function. Find the minimizer and the minimum value and indicate them on the sketch.
   (a) $\min_{x,y} |x| + |y|$ subject to $y = 3x - 2$.
   (b) $\min_{x,y} \max(|x|, |y|)$ subject to $y = 3x - 2$.

6. *Legendre Transform/Conjugate functions* The formula for the Legendre transform $f^*(y)$ of the function $f(x)$ is given by
$$f^*(y) = \max_x \Big(xy - f(x) \Big)$$
   (a) Compute the Legendre Transform $f^*(y)$ of $f(x) = 3x^2 + 5$.
   (b) Compute the Legendre Transform $f^*(y)$ of $f(x) = e^x + 4x$.
   (c) For (b), find the Legendre Transform $f^{**}(x)$ of $f^*(y)$ defined as
$$f^{**}(x) = \max_y \Big(xy - f^*(y) \Big)$$
   Is is related to $f(x)$?

7. *Projection in different norms* Let $x_0 = (4, 0)$. Consider the problem

$$\begin{array}{ll} \min & f(x) = \|x - x_0\|_p, \\ \text{subject to} & x \in L = \{(x_1, x_2) \mid x_2 = 5x_1\} \end{array}$$

   (a) Find the minimizer x_* for for $p = 1, 2, \infty$.

   (b) Make a sketch illustrating the point x_0, the line L, and the shape (circle, square, and diamond) of the norm about x_0 which intersects the line.

8. Consider the problem

$$\begin{array}{ll} \min & f(x) = \dfrac{1}{2}x_1^2 + \dfrac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 1. \end{array}$$

   (a) Find the minimizer $x(\mu)$ and the estimated Lagrange Multiplier $\lambda(\mu)$ of the logarithmic barrier method.

   (b) Find x^* and x^* by taking the limit $\mu \to 0$.

9. Consider the following problem

$$\min\ f(x), \qquad f(x) = (x_1 + 1)^2 + (x_2 - 1)^2 + |x_1 + x_2|.$$

   (a) Is this problem convex?

   (b) The function f is unconstrained, but the term $|x_1 + x_2|$ is not differentiable. We can get rid of the absolute value by introducing $x_3 = |x_1 + x_2|$, and get the equivalent one

$$\min_{x \in \Omega} \tilde{f}(x) = (x_1 + 1)^2 + (x_2 - 1)^2 + x_3.$$

   Write down the **linear inequality** constraints characterizing the feasible region.

10. Consider the problem $\min_{x \in \mathbb{R}^2} f(x)$ where

$$f(x) = (x_1 + 1)^2 + (x_2 - 1)^2 + \|x\|_1 = (x_1 + 1)^2 + (x_2 - 1)^2 + |x_1| + |x_2|.$$

   (a) Introducing $x_3 = |x_1|, x_4 = |x_2|$, formulate this problem as a constrained optimization problem with **linear inequality** constraints (similar to the previous one).

   (b) Determine which quadrant the global minimizer (x_1, x_2) will be. Use this information to get rid of the absolute value (so that you can take derivatives) and find the global minimizer.