Continuous Optimization

Penalty and Barrier

Sections covered in the textbook (2nd edition):

- Chapter 17: **1, 2, 3**

Barrier Method

$$\min_{x \in \Omega} \qquad f(x)$$

where Ω is the feasible region

$$\Omega = \{x \mid c_i(x) = 0, i \in \mathcal{E},\ c_i(x) \geq 0, i \in \mathcal{I}\}$$

This is equivalent to

$$\min f(x) + \chi_\Omega(x)$$

where $\chi_\Omega(x)$ is the indicator function,

$$\chi_\Omega(x) = \begin{cases} 0, & \text{if } x \in \Omega, \\ \infty, & \text{otherwise.} \end{cases}$$

But we have to get alternative formulas for $\chi_\Omega(x)$ to make it useful in practice, like the duality theory.

Barrier: Logarithmic and inverse function

$$\begin{array}{ll} \min & f(x), \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, 2, \cdots, m. \end{array}$$

Introducing the barrier function

$$\beta_\mu(x) = f(x) + \rho\phi(x)$$

such that $\phi(x) \to +\infty$ as $g_i(x) \to 0^+$, with the corresponding minimizer x_ρ^*. Then find the limit $\rho \to 0^+$.

Two popular barrier functions: the logarithmic function

$$\phi(x) = -\sum_{i=1}^{m} \ln g_i(x),$$

and inverse function

$$\phi(x) = \sum_{i=1}^{m} \frac{1}{g_i(x)}.$$

Barrier methods

Solve the following problem using logarithmic barrier

$$\begin{aligned} \min \quad & f(x) = x_1 - 2x_2, \\ \text{subject to} \quad & 1 + x_1 - x_2^2 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

Another example

$$\begin{aligned} \min \quad & f(x) = x_1^2 + x_2^2, \\ \text{subject to} \quad & x_1 - 1 \geq 0 \geq 0, \\ & x_2 + 1 \geq 0. \end{aligned}$$

Show that the result converges and we can get an estimate of the Lagrange Multipliers.
Also show that the Hessian matrix is ill-conditioned ($\lambda_{\max}/\lambda_{\min}$ is large).

Penalty method

$$\min_{x} f(x) \qquad \text{subject to} \quad c_i(x) = 0, \quad i \in \mathcal{E}.$$

Equivalent to $\min f(x) + \mu\psi(x)$, such that $\psi(x) = 0$ if x is feasible and $\psi(x) > 0$ otherwise.

$$Q(x;\mu) \stackrel{\text{def}}{=} f(x) + \frac{\mu}{2} \sum_{i\in\mathcal{E}} c_i^2(x).$$

For general constrained problems

$$\min_{x} f(x) \qquad \text{subject to} \quad c_i(x) = 0,\ i \in \mathcal{E},\ c_i(x) \geq 0, i \in \mathcal{I}.$$

the quadratic penalty function is

$$Q(x;\mu) \stackrel{\text{def}}{=} f(x) + \frac{\mu}{2} \sum_{i\in\mathcal{E}} c_i^2(x) + \frac{\mu}{2} \sum_{i\in\mathcal{I}} ([c_i(x)]^-)^2.$$

Nonsmooth penalty functions

For general constrained problems

$$\min_x f(x) \qquad \text{subject to} \qquad c_i(x) = 0,\ i \in \mathcal{E},\ c_i(x) \geq 0, i \in \mathcal{I}.$$

the nonsmooth penalty function is

$$Q(x; \mu) \stackrel{\text{def}}{=} f(x) + \mu \sum_{i \in \mathcal{E}} |c_i(x)| + \mu \sum_{i \in \mathcal{I}} [c_i(x)]^{-}.$$

Penalty method

$$\begin{array}{ll} \min & f(x) = -x_1 x_2, \\ \text{subject to} & x_1 + 2x_2 - 4 = 0. \end{array}$$

Define the penalty function

$$Q(x;\mu) = -x_1 x_2 + \mu(x_1 + 2x_2 - 4)^2.$$

- Find the minimizer x_μ^*
- How far $c(x_\mu^*)$ away from zero?
- Estimate the Lagrange Multiplier
- Check the condition number of $\nabla^2 Q(x_\mu^*)$