Continuous Optimization

Linear Inequality Constrained Optimization

Sections covered in the textbook (2nd edition):

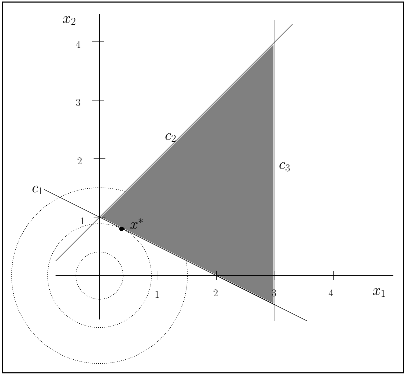
- Chapter 12 (linear inequality constrained problems)

Suggested exercises in the textbook:

- 12.14, 12.15

Linear inequality constraints

$$
\begin{aligned}
\text{min} \quad & f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\
\text{subject to} \quad & x_1 + 2x_2 \geq 2 \\
& x_1 - x_2 \geq -1 \\
& -x_1 \geq -3.
\end{aligned}
$$



(a) What's the *active set*, according to the graph?

(b) Solve the problem subject inequality constraint(s) in the active set, using Lagrange Multiplier.

(c) If x is close to x^*, but still in the feasible region, how does the objective function change?

(d) How about the Lagrange Multipliers with the rest *inactive set*?

Linear inequality constraints

The same previous problem, what if we erroneously guessed only the third constraint $-x_1 \geq -3$ was active?

(i) Find the minimizer $\tilde{x}^*$ under this (wrong) active constraint.

(ii) Find a feasible direction at $\tilde{x}^*$ along which the objective function is decreasing.

(iii) Find the Lagrange Multiplier

Repeat with the wrong active constraint $x_1 - x_2 \geq -1$.

Optimality conditions

Original problem

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & Ax \geq b. \end{array}$$

If x^* is a local minimizer, take the active constraints $a_i^t x^* = b_i$ or $\hat{A}x^* = b$. Then x^* is a local minimizer of the problem

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & \hat{A}x = \hat{b}. \end{array}$$

The **optimality conditions** are exactly the one for the above equality constrained problem.

Optimality conditions

The equivalent (at least for optimality conditions) equality constrained problems

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & \hat{A}x = \hat{b}. \end{array}$$

The solutions $x = \bar{x} + Zv$ where Z is a null-space matrix of $\hat{A}$.
Two equivalent **first order necessary conditions**:

$$Z^t \nabla f(x^*) = 0 \quad \Longrightarrow \quad \nabla f(x^*) = \hat{A}^t \hat{\lambda}^*$$

and the **second-order necessary conditions**:

$$Z^t \nabla^2 f(x^*) Z \text{ is nonnegative definite.}$$

The sign of $\hat{\lambda}^*$?

Optimality conditions

If the Lagrange multipliers for the rest of constraints are zeros, then we have *complementary slackness conditions*

$$\lambda_i^*(a_i^t x^* - b_i) = 0$$

Theorem (Necessary condition for linear ineq constr)

If x^* is a local minimizer of f over the set $\{x : Ax \geq b\}$, then for some vector λ^* of Lagrange multipliers,

- $\nabla f(x^*) = A^T \lambda^*$ *(or $Z^t \nabla f(x^*) = 0$)*
- $\lambda^* \geq 0$
- ${\lambda^*}^t (Ax^* - b) = 0$
- $Z^t \nabla f(x^*) Z$ *is nonnegative definite.*

The sufficient condition needs a little bit more than $Z^t \nabla f(x^*) Z$ *is positive definite*, for *degenerate* constraints.

Example for Sufficient conditions

$$\begin{aligned} \min \quad & f(x) = x_1^3 + x_2^2 \\ \text{subject to} \quad & -1 \le x_1 \le 0. \end{aligned}$$

If $a_i^t x^* = b_i$, there may be problem if $\lambda_i^* = 0$ (the complementary slackness conditions are not **strict**).

Combinatorial Complex by selecting active sets

Solve the following problem by selecting different combination of active sets.

$$
\begin{aligned}
\min \quad & f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\
\text{subject to} \quad & x_1 + 2x_2 \geq 2 \\
& x_1 - x_2 \geq -1 \\
& -x_1 \geq -3.
\end{aligned}
$$