Optimality conditions for nonlinear constraints

$$
\begin{aligned}
\min \quad & f(x) \\
\text{subject to} \quad & c_i(x) = 0, \quad i \in \mathcal{E}, \\
& c_i(x) \geq 0, \quad i \in \mathcal{I}.
\end{aligned}
$$

Theorem (First-order necessary (KKT) conditions)

Suppose x^* is a local solution, f and c_i are continuously differentiable and the **LICQ** *holds at x^*. Then there exist a Lagrange multiplier λ^*, $i \in \mathcal{E} \bigcup \mathcal{I}$, such that*

(1) $c_i(x^*) = 0,$ *for all* $i \in \mathcal{E}$ (*Feasible condition*)

(2) $c_i(x^*) \geq 0,$ *for all* $i \in \mathcal{I}$ (*Feasible condition*)

(3) $\lambda_i^* \geq 0,$ *for all* $i \in \mathcal{I}$

(4) $\lambda_i^* c_i(x^*) = 0,$ *for all* $i \in \mathcal{E} \bigcup \mathcal{I}$ (*Complementarity*)

(5) $\nabla_x L(x^*, \lambda^*) = 0$

Optimality conditions for nonlinear constraints

The critical cone

$$\mathcal{C}(x^*, \lambda^*) = \{w \in \mathcal{F}(x^*) | \nabla c_i(x^*)^t w = 0, \quad \forall\ i \in \mathcal{A}(x^*) \bigcap \mathcal{I} \text{ with } \lambda_i^* > 0\}$$

Theorem (Second-order necessary conditions)

$$w^T \nabla_{xx} L(x^*, \lambda^*) w \geq 0, \qquad \forall\ w \in \mathcal{C}(x^*, \lambda^*).$$

Theorem (Second-order sufficient conditions)

Strict Complementarity $+$

$$w^T \nabla_{xx} L(x^*, \lambda^*) w > 0, \qquad \forall\ w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0.$$

Is this consistent with linear constraints?

Example 1:

$$\min \; x_1 + x_2 \qquad \text{subject to} \;\; 2 - x_2^2 - x_2^2 \geq 0.$$

Example 2:

$$\begin{array}{ll} \min & f(x) = x_1 \\ \text{subject to} & (x_1 + 1)^2 + x_2^2 \geq 1 \\ & x_1^2 + x_2^2 \leq 2. \end{array}$$

What happens if one of the constraint $\nabla c_i(x^*) = 0$ (*irregular point*)?

$$\begin{array}{ll} \min & x \\ \text{subject to} & x^3 \geq 0. \end{array}$$

Counterexample for LICQ

$$\begin{aligned} \text{min} \quad & f(x) = 3x_1 + 4x_2 \\ \text{subject to} \quad & (x_1+1)^2 + x_2^2 = 1 \\ & (x_1-1)^2 + x_2^2 = 1 \end{aligned}$$

(i) Find the feasible region and the minimizer.

(ii) Can you find λ^*?

(iii) How about one of the constraint is perturbed a little, say c_1 becomes

$$(x_1+1)^2 + x_2^2 = 1 + \delta.$$

Continuous Optimization

Duality

Sections covered in the textbook (2nd edition):

- Chapter 12, Section 8 and 9

Suggested exercises in the textbook:

- 12.22

Lagrange Multipliers and sensitivity

Facts:

(a) Complementarity: For inactive constraints $c_i(x^*) \geq 0$, the corresponding Lagrange Multiplier $\lambda_i^* = 0$.

(b) If the constraint $c_i(x) \geq 0$ (or $c_i(x) = 0$) is perturbed to $c_i(x) \geq \delta$ (or $c_i(x) = \delta$) then the Lagrange function evaluated at the optimal x_δ^* and λ_δ^* has the relation

$$\frac{d}{d\delta} L_\delta(x_\delta^*, \lambda_\delta^*) \bigg|_{\delta=0} = \lambda_i^*$$

or

$$L_\delta(x_\delta^*, \lambda_\delta^*) = L(x^*, \lambda^*) + \lambda_i^* \delta + O(\delta^2)$$

(c) The inequality constraint c_i is *strongly active* if $i \in \mathcal{A}(x^*)$ and $\lambda_i^* > 0$. It is *weakly active* if $i \in \mathcal{A}(x^*)$ and $\lambda_i^* = 0$.

General min-max duality

$$\min_{x \in X} \max_{y \in Y} \mathcal{F}(x, y) \qquad \text{vs} \qquad \max_{y \in Y} \min_{x \in X} \mathcal{F}(x, y)$$

If both have solution in the sense that

$$\max_{y \in Y} \min_{x \in X} \mathcal{F}(x, y) = \min_{x \in X} \mathcal{F}(x, y^*),$$

$$\min_{x \in X} \max_{y \in Y} \mathcal{F}(x, y) = \max_{y \in Y} \mathcal{F}(x^*, y)$$

then we have the **Weak duality**:

$$\max_{y \in Y} \min_{x \in X} \mathcal{F}(x, y) \leq \min_{x \in X} \max_{y \in Y} \mathcal{F}(x, y)$$

This implies the *saddle point condition*:

$$\mathcal{F}(x^*, y) \leq \mathcal{F}(x^*, y^*) \leq \mathcal{F}(x, y^*).$$

General min-max duality

Strong duality: The condition

$$\max_{y \in Y} \min_{x \in X} \mathcal{F}(x,y) \leq \min_{x \in X} \max_{y \in Y} \mathcal{F}(x,y)$$

holds if and only if there exists a pair (x^*, y^*) that satisfies the saddle point condition for $\mathcal{F}$.

Example (Two-person Zero-sum Game represented as matrix).

	B chooses B1	B chooses B2	B chooses B3
A chooses A1	+3	-2	+2
A chooses A2	-1	0	+4
A chooses A3	-4	-3	+1

Lagrange Duality for $L(x, \lambda) = f(x) - \lambda^t c(x)$

For the problem

$$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{subject to } c(x) \geq 0,$$

it is equivalent to $\min_x L^*(x)$ (if the feasible region is not empty), where

$$L^*(x) = \max_{\lambda \geq 0} L(x, \lambda) = \begin{cases} f(x), & \text{if } g(x) \geq 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Define the **dual objective function** q as

$$q(\lambda) \stackrel{\text{def}}{=} \inf_x L(x, \lambda)$$

The **dual problem**:

$$\max_{\lambda \in \mathbb{R}^m} q(\lambda) \qquad \text{subject to } \lambda \geq 0.$$

Lagrange Duality for $L(x, \lambda) = f(x) - \lambda^t c(x)$

Example: Find the dual problem for

$$\min_{x \in \mathbb{R}^2} \frac{1}{2}(x_1^2 + x_2^2) \qquad \text{subject to } x_1 - 1 \geq 0.$$

Theorem

The function q is concave.

Theorem (Weak duality)

If $\bar{x}$ is feasible and any $\bar{\lambda} \geq 0$, we have $q(\bar{\lambda}) \leq f(\bar{x})$.

Duality and optimality conditions

If f and $-c_i$ are convex and differentiable at x^*, (x^*, λ^*) satisfies the first order necessary (KKT) condition.

(a) λ^* is also a solution of the dual problem

$$\max_{\lambda \in \mathbb{R}^m} q(\lambda) \qquad \text{subject to } \lambda \geq 0.$$

(b) If LICQ holds at x^* and $\hat{\lambda}$ solves the dual problem with the infimum of $L(x, \hat{\lambda})$ is attained at $\hat{x}$ and $L(x, \hat{\lambda})$ is strictly convex in x. Then $\hat{x} = x^*$ and $f(x^*) = L(\hat{x}, \hat{\lambda})$.

(c) (Wolfe duality) If (x^*, λ^*) is a solution of the primary problem

$$\min_{x \in \mathbb{R}^n} f(x) \qquad \text{subject to } c(x) \geq 0,$$

then it solves

$$\begin{aligned} \max_{x,\lambda} \quad & L(x, \lambda) \\ \text{subject to} \quad & \nabla_x L(x, \lambda) = 0, \lambda \geq 0. \end{aligned}$$

Other general examples

Linear Programming:

$$\min c^t x, \qquad \text{subject to } Ax - b \geq 0.$$

Convex Quadratic programming

$$\min \frac{1}{2} x^t Q x + c^t x \qquad \text{subject to} Ax - b \geq 0,$$

where Q is positive definite.