Sufficient condition for linear ineq constraint I

We need **positive definite** instead of nonnegative definite (or positive semidefinite) as that for unconstrained or linear equality constrained problems; the extra condition here is **strict complementarity** (or non-degeneracy) at the point.

Theorem (Sufficient Condition 1)

If x^* satisfies

- $Ax^* \geq b$
- $\nabla f(x^*) = A^t \lambda^*$
- $\lambda^* \geq 0$
- *Strict complementarity holds*
- $Z^t \nabla^2 f(x^*) Z$ *is positive definite,*

then x^* is a strict local minimizer for the problem

$$\min f(x) \quad \textit{subject to} \quad Ax \geq b.$$

Sufficient condition for linear ineq constraint II

Alternatively we can choose Z differently by avoiding those degenerate constraints.

Theorem (Sufficient Condition 1)

Let $\hat{A}_+$ be the submatrix of $\hat{A}$ corresponding to the non-degenerate active constraints at x^* (those constraints whose Lagrange Multiplier are positive). Let Z_+ be a basis matrix for the null space of $\hat{A}_+$. If x^* satisfies

- $Ax^* \geq b$
- $\nabla f(x^*) = A^t \lambda^*$
- $\lambda^* \geq 0$
- $Z_+^t \nabla^2 f(x^*) Z_+$ *is positive definite,*

then x^* is a strict local minimizer for the problem

$$\min f(x) \quad \text{subject to} \quad Ax \geq b.$$

Sufficient condition for linear ineq constraint

Show the problem

$$\begin{aligned} \min \quad & f(x) = x_1^3 + x_2^2 \\ \text{subject to} \quad & -1 \le x_1 \le 0. \end{aligned}$$

does not satisfy the sufficient condition at $(0, 0)$.

Solve the following problem:

$$\begin{aligned} \min \quad & f(x) = x_1^3 - x_2^3 - 2x_1^2 - x_1 + x_2 \\ \text{subject to} \quad & -x_1 - 2x_2 \ge -2, \\ & x_1 \ge 0, \\ & x_2 \ge 0. \end{aligned}$$

Modification with the presence of equality

Solve the previous with the first one with equality in two ways:

$$
\begin{aligned}
\min \quad & f(x) = x_1^3 - x_2^3 - 2x_1^2 - x_1 + x_2 \\
\text{subject to} \quad & -x_1 - 2x_2 = -2, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & f(x) = x_1^3 - x_2^3 - 2x_1^2 - x_1 + x_2 \\
\text{subject to} \quad & x_1 + 2x_2 = 2, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned}
$$

Continuous Optimization

Nonlinear Constrained Optimization

Sections covered in the textbook (2nd edition):

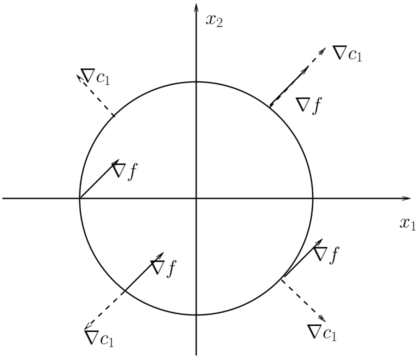
- Chapter 12 (Nonlinear constrained problems)

Suggested exercises in the textbook:

- 12.11, 12.13, 12.15, 12.18, 12.19, 12.21

Nonlinear Equality Constraints

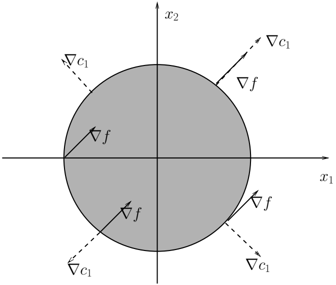
$$\texttt{min } f(x) = x_1 + x_2 \qquad \texttt{s.t.} \quad c(x) = x_1^2 + x_2^2 - 2 = 0.$$



At the minimizer x^*, there is no ("feasible") direction d s.t. $d^t \nabla c(x^*) = 0$ and $d^t \nabla f(x^*) < 0 \Longrightarrow \nabla f(x^*) = \lambda \nabla c(x^*)$.

Nonlinear Inequality constraint

$$\texttt{min } f(x) = x_1 + x_2 \qquad \texttt{s.t.} \quad c(x) = 2 - x_1^2 - x_2^2 \geq 0.$$

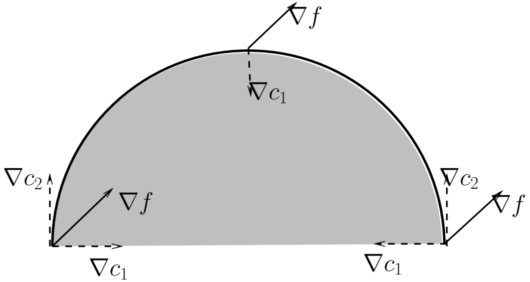


What's the difference for the two cases: $c(x^*) < 0$ and $c(x^*) = 0$? Any modification to the condition $\nabla f(x^*) = \lambda \nabla c(x^*)$?

How about with the constraint $c(x) = x_1^2 + x_2^2 - 2 \leq 2$?

More inequality constraints

$$\begin{array}{ll} \min & f(x) = x_1 + x_2 \\ \text{s.t.} & c_1(x) = 2 - x_1^2 - x_2^2 \geq 0 \\ & c_2(x) = x_2 \geq 0. \end{array}$$



At the minimizer x^*,

$$\nabla f(x^*) = \lambda_1^* \nabla c_1(x^*) + \lambda_1^* \nabla c_1(x^*), \qquad \lambda_i^* c_i(x^*) = 0, i = 1, 2.$$

Not true for any other points in the feasible region.

Tangent cone and Constraint qualifications

The vector d is a **tangent** (or **tangent vector**) to Ω at a point x if there are a feasible sequence $\{z_k\}$ approaching x and a sequence of positive scalars $\{t_k\}$ with $t_k \to 0$ such that

$$\lim_{k\to\infty} \frac{z_k - x}{t_k} = d$$

The set of all tangents to Ω at x^* is called the **tangent cone** and is denoted by $T_\Omega(x^*)$.

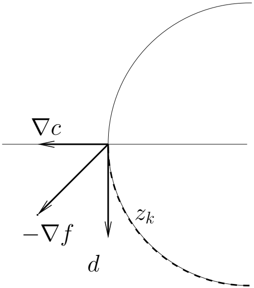
The set of **linearized feasible direction** $\mathcal{F}(x)$ at a feasible point x

$$\mathcal{F}(x) = \left\{ d \;\middle|\; \begin{array}{l} d^t \nabla c_i(x) = 0, \text{ for all } i \in \mathcal{E} \\ d^t \nabla c_i(x) \geq 0, \text{ for all } i \in \mathcal{A}(x) \bigcap \mathcal{I} \end{array} \right\}$$

These concepts are introduced to investigate the behavior of f near x^*.

Approaching non-optimal point

$$\min\ f(x) = x_1 + x_2 \qquad \texttt{s.t.}\ \ c(x) = x_1^2 + x_2^2 - 2 = 0.$$

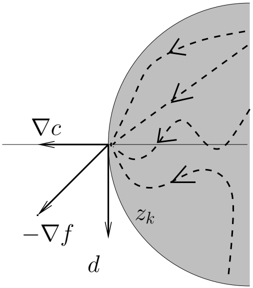


Find $T_\Omega(x)$ and $\mathcal{F}(x)$. How about the constraint becomes the equivalent one $c(x) = (x_1^2 + x_2^2 - 2)^2 = 0$?

The point $x = (-\sqrt{2}, 0)$ is not optimal because there exists a feasible sequence $\{z_k\}$ such that $f(z_k) < f(x)$.

Approaching non-optimal point

$$\min f(x) = x_1 + x_2 \qquad \texttt{s.t.} \quad c(x) = 2 - x_1^2 - x_2^2 \geq 0.$$



What's $T_\Omega(x)$ and $\mathcal{F}(x)$ at $x = (-\sqrt{2}, 0)$?

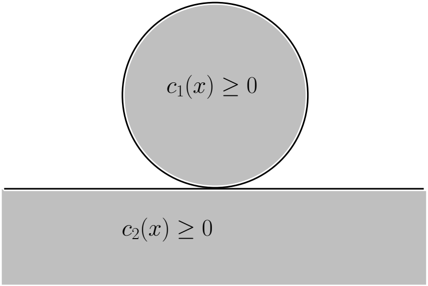
Constraint qualifications: The geometry of the feasible region is **well described** by the algebraic quantities by c_i, for example no constraint like $x_1^3 \geq 0$.

LICQ

For the constraints

$$c_1(x) = 1 - x_1^2 - (x_2 - 1)^2 \geq 0, \qquad c_2(x) = -x_2 \geq 0,$$

check that $T_\Omega(x) \neq \mathcal{F}(x)$ at $x = (0, 0)$.



Linear Independence constraint qualification (LICQ) holds at a point x if the set of active constraint gradients $\{\nabla c_i(x), i \in \mathcal{A}(x)\}$ is linearly independent.