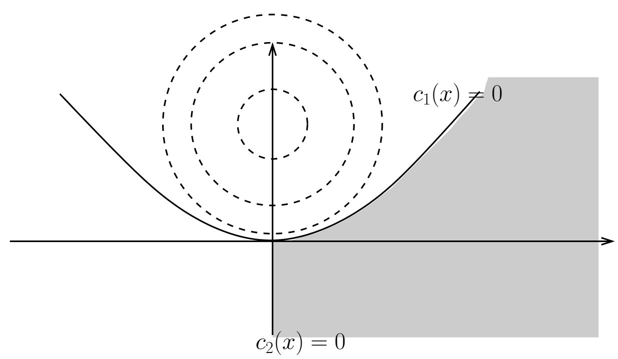
Solution to Midterm 2

1. Consider the problem

$$\begin{array}{ll} \min & f(x) = x_1^2 + (x_2-3)^2 \\ \text{subject to} & c_1(x) = x_1^2 - 2x_2 \geq 0, \\ & c_2(x) = x_1 \geq 0. \end{array}$$

(a) Plot the feasible region and three contours of the objective function (No need to find the minimizer, because it depends on how accurately you draw them).



(b) Given that c_2 is NOT active at the global minimizer x^*, find x^*.

If c_2 is not active at the global minimizer x^*, then $\lambda_2^* = 0$ and the Lagragian (or Lagrange function) can be written as

$$L(x, \lambda) = f(x) - \lambda_1 c_1(x) = x_1^2 + (x_2-3)^2 - \lambda_1(x_1^2 - 2x_2).$$

The minimizer x^* and the Lagrangian Multiplier λ_1^* at that point satisfy

$$\nabla_x L(x, \lambda) = \begin{pmatrix} 2x_1^* - 2\lambda_1^* x_1^* \\ 2x_2^* - 6 + 4\lambda_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the constraint $c_2(x) = x_1 \geq 0$ is *inactive*, we have $x_1^* \neq 0$ and from the first equation $\lambda_1^* = 1$. Substituting it into the second equation, $x_2^* = 2$. Using the active constraint $c_1(x) = x_1^2 - 2x_2 = 0$, we get $x_1^* = 2$ (only keep those roots in the feasible region, i.e., $x_1^* \geq 0$). Therefore, the unique (global) minimizer is $x^* = (2, 2)$.

(c) Show that the **second order sufficient condition** is satisfied at x^* (no need to check the first order conditions).

At $x^* = (2, 2)$, we have for the only active constraint c_1, $\nabla c_1(x^*) = (2x_1^*, -2)^t = (4, -2)^t$. Any vector $w = (d_1, d_2)^t$ in the critical cone $\mathcal{C}(x^*, \lambda^*)$ if and only if $w^t \nabla c(x^*) = 0$. Hence $4d_1 - 2d_2 = 0$ or $w = (d, 2d)^t$ and $\mathcal{C}(x^*, \lambda^*) = \{(d, 2d)^t, d \in \mathbb{R}\}$.

For $w = (d, 2d)^t = \mathcal{C}(x^*, \lambda^*)$,

$$w^t \nabla^2 L(x^*, \lambda^*) w = (d, 2d) \begin{pmatrix} 2 - 2\lambda_1^* & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} d \\ 2d \end{pmatrix} = 8d^2 > 0$$

if $w \neq 0$. Therefore, the second order sufficient condition is satisfied.

2. Consider the problem

$$\begin{aligned} \min \quad & f(x) = x_1^2 + (x_2-1)^2 \\ \text{subject to} \quad & c(x) = x_1^2 - \kappa x_2 \geq 0. \end{aligned}$$

Here κ is a positive constant. Find the critical κ_c, such that $(0,0)$ is a local minimizer for any $\kappa > \kappa_c$ (and $(0,0)$ is not a local minimizer when $\kappa < \kappa_c$).

The Lagrangian is $L(x,\lambda) = f(x) - \lambda c(x) = x_1^2 + (x_2-1)^2 - \lambda(x_1^2 - \kappa x_2)$ and the any local minimizer satisfies the necessary condition

$$\nabla_x L(x^*, \lambda^*) = \begin{pmatrix} 2x_1^* - 2\lambda^* x_1^* \\ 2x_2^* - 2 + \kappa\lambda^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

At the local minimizer $x^* = (0,0)$, the first equation is satisfied and from the second equation $\kappa\lambda^* = 2$. We have to use the second order condition to find the critical value κ_c to be a local minimizer. The vector $w = (d_1, d_2)^t$ in the critical cone $\mathcal{C}(x^*, \lambda^*)$ if and only if $w^t \nabla c(x^*) = -\kappa d_2 = 0$. This implies that $d_2 = 0$ and $\mathcal{C}(x^*, \lambda^*) = \{(d,0)^t, d \in \mathbb{R}\}$. The point $x^* = (0,0)$ is a local minimizer if for $w = (d,0)^t \in \mathcal{C}(x^*, \lambda^*)$,

$$w^t \nabla^2 L(x^*, \lambda^*) w = (d, 0) \begin{pmatrix} 2 - 2\lambda^* & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} = (2 - 2\lambda^*) d^2.$$

Therefore, $x^* = (0,0)$ is a local minimizer if $2 - 2\lambda^* > 0$ (second order sufficient condition) and it is not a local minimizer if $2 - 2\lambda^* < 0$ (second order necessary condition). The critical value for λ^* is $\lambda_c^* = 1$ and the critical value $\kappa_c = 2/\lambda_c^* = 2$.

3. Write down the dual problem of the following linear programming

$$\begin{aligned} \min \quad & f(x) = x_1 \\ \text{subject to} \quad & x_1 + x_2 = 1, \\ & x_2 \leq 0, \\ & x_2 \geq -1. \end{aligned}$$

The primal problem is equivalent to

$$\min_x \max_{\lambda_2 \geq 0, \lambda_3 \geq 0} x_1 - \lambda_1(x_1 + x_2 - 1) + \lambda_2 x_2 - \lambda_3(x_2 + 1).$$

There is no constraint on λ_1 because of the equality $x_1 + x_2 = 1$ and the sign in front of the term $\lambda_2 x_2$ is positive because the original constraint is $x_2 \leq 0$ (not in "$\geq$").

The dual problem is obtained by interchange the order of min and max, i.e.,

$$\max_{\lambda_2 \geq 0, \lambda_3 \geq 0} \min_x \; x_1 - \lambda_1(x_1 + x_2 - 1) + \lambda_2 x_2 - \lambda_3(x_2 + 1) = \max_{\lambda_2 \geq 0, \lambda_3 \geq 0} q(\lambda),$$

where

$$q(\lambda) = \min_x \; x_1 - \lambda_1(x_1 + x_2 - 1) + \lambda_2 x_2 - \lambda_3(x_2 + 1).$$

Since there is no constraint on x, $q(\lambda)$ is finite if and only iff the coefficient of x_1 and x_2 are both zero (equivalently the gradient w.r.t. x is zero as in the general cases). From

$$x_1 - \lambda_1(x_1 + x_2 - 1) + \lambda_2 x_2 - \lambda_3(x_2 + 1) = \lambda_1 - \lambda_3 + x_1(1 - \lambda_1) + x_2(-\lambda_1 + \lambda_2 - \lambda_3),$$

we have the constraints on λ:

$$1 - \lambda_1 = 0, \qquad -\lambda_1 + \lambda_2 - \lambda_3 = 0$$

and the dual problem (you can further simplify it, but not required) is

$$\begin{aligned} \min \quad & 1 - \lambda_3 \\ \text{subject to} \quad & 1 - \lambda_1 = 0, \\ & -\lambda_1 + \lambda_2 - \lambda_3 = 0, \\ & \lambda_2 \geq 0, \; \lambda_3 \geq 0. \end{aligned}$$