Solution to Midterm 1

1. (10 pt) Find the **largest connected interval** on which $f(x) = 1/(1+x^2)$ is convex.

Since f is smooth, it is convex on some domain if and only if $f''(x) \geq 0$ on that domain. It is easy to see that

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

$$f''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3}.$$

f is convex on the domain where $6x^2 - 2 \geq 0$ or $|x| > 1/\sqrt{3}$. Therefore the largest connected interval on which f is convex is $[1/\sqrt{3}, \infty)$.

2. (20 pt)Let $f(x,y) = x^2 + y^2 - xy$.

(a) (5 pt)Compute $\nabla f(x), Hf(x)$. Is f convex? Explain your answer.

$$\nabla f = \begin{pmatrix} 2x-y \\ 2y-x \end{pmatrix}, \quad \nabla^2 f = Hf = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

It is easy to see that the two eigenvalues of Hf $\lambda_1 = 1, \lambda_2 = 3$ are both positive. This implies that Hf is positive definite and f is covex. (You can also use the fact that $\mathrm{tr}(Hf) = 4 > 0$ and $\det(Hf) = 3 > 0$ to show Hf is positive definite, or two positive eigenvalues).

(b) (5 pt)Find the minimizer of $f(x,y)$.

The equation $\nabla f = 0$ has the solution $(x^*, y^*) = (0,0)$. Since f is convex, this is the unique (global) minimizer.

(c) (5 pt)Write out the formula for the Steepest Descent method for function minimization.

The iterative scheme for Steepest Descent method is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \alpha_n \vec{p}_n$$

where $\vec{p}_n$ is the negative gradient direction

$$\vec{p}_n = -\nabla f(x_n, y_n) = \begin{pmatrix} y_n - 2x_n \\ x_n - 2y_n \end{pmatrix}$$

and α_n is the optimal length to minimize $\phi(\alpha) = f((x_n, y_n) + \alpha\vec{p}_n) = f((1-2\alpha)x_n + \alpha y_n, (1-2\alpha)y_n + \alpha x_n)$.

(d) (5 pt)Compute one Steepest Descent iteration, starting with initial point $(x^0, y^0) = (1, 1)$.

Start with $(x_0, y_0) = (1, 1)$, we have $\vec{p}_0 = (-1, -1)$ and

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \alpha\vec{p}_0 = \begin{pmatrix} 1-\alpha \\ 1-\alpha \end{pmatrix}$$

Here α is the minimizer of

$$\phi(\alpha) = f(x_1, y_1) = f(1-\alpha, 1-\alpha) = (1-\alpha)^2,$$

which is given by $\phi'(\alpha_1) = 0$ or $\alpha_0 = 1$. Therefore $(x_1, y_1) = (0, 0)$.

Remark. In general, $\alpha \neq 1$ and you can not get to the minimizer in just one step. Try $(x_0, y_0) = (0, 1)$ to see this.

3. (20 pt)Let $\delta > 0$, x_0 be a real number and the function $f(x)$ is defined as

$$f(x) = \frac{1}{2}(x - x_0)^2 + \delta|x|.$$

(a) (8 pt) Is f convex? Why?

Let $f_1(x) = (x - x_0)^2/2$ and $f_2(x) = |x|$. Since both f_1 ($f_1'' \geq 0$) and f_2 (which is a norm) are convex, so is their sum.

(b) (12 pt)Find the global minimizer of $f(x)$.

When $x \geq 0$, $f(x) = (x - x_0)^2/2 + \delta x$ and $f'(x) = x - x_0 + \delta$. If $x_0 - \delta \geq 0$ then the minimizer is at $x^* = x_0 - \delta$, otherwise $f'(x) > 0$ for $x \geq 0$ and the minimizer is at $x^* = 0$.

When $x \leq 0$, $f(x) = (x - x_0)^2/2 - \delta x$ and $f'(x) = x - x_0 - \delta$. If $x_0 + \delta < 0$ then the minimizer is at $x^* = x_0 + \delta$, otherwise $f'(x) < 0$ for $x \leq 0$ and the minimizer is at $x^* = 0$.

Put all these together, we have the global minimizer

$$x^* = \begin{cases} x_0 - \delta, & \text{if } x_0 \geq \delta, \\ 0, & \text{if } -\delta < x_0 < \delta, \\ x_0 + \delta, & \text{if } x_0 \leq -\delta. \end{cases}$$