Math 309 **Spring 2012**

Continuous Optimization

Final Exam April 15, 2012

Name (print): ______________________

Computing ID: ______________________

Signature: ______________________

Question	Grade
1	
2	
3	
4	
5	
6	
7	
8	
Total (out of 100)	

Guidelines

(i) Read the problems carefully. The statement for some of the problems can be long, but they just provide a little more background information (not correlated to the difficulty of the problem in any way).

(ii) This is a long final exam. Only write down what you are asked to do (with some details). For example, if you are just asked to find the "global minimizer", then find it and do not check optimality conditions (unless you are asked).

(iii) The problems are NOT ordered according to their difficulties.

(iv) Use the hint wisely and use the symmetry of the problem or equations to reduce the amount of work .

(v) You don't need a calculator. The focus here is to understand the definitions, concepts and the connections between them, but not complicated calculation.

Theorem 0.1 (First-order necessary (KKT) conditions). *Suppose x^* is a local solution, f and c_i are continuously differentiable and the* **LICQ** *holds at x^*. Then there exist a Lagrange multiplier λ^*, $i \in \mathcal{E} \bigcup \mathcal{I}$, such that*

(1) $c_i(x^*) = 0, \quad$ *for all* $i \in \mathcal{E}$ *(Feasible condition for equality constraints)*

(2) $c_i(x^*) \geq 0, \quad$ *for all* $i \in \mathcal{I}$ *(Feasible condition for inequality constraints)*

(3) $\lambda_i^* \geq 0, \quad$ *for all* $i \in \mathcal{I}$

(4) $\lambda_i^* c_i(x^*) = 0, \quad$ *for all* $i \in \mathcal{E} \bigcup \mathcal{I}$ *(Complementarity)*

(5) $\nabla_x L(x^*, \lambda^*) = 0$

Theorem 0.2 (Second-order necessary conditions).

$$w^T \nabla_{xx} L(x^*, \lambda^*) w \geq 0, \qquad \forall\ w \in \mathcal{C}(x^*, \lambda^*).$$

Theorem 0.3 (Second-order sufficient conditions).

$$w^T \nabla_{xx} L(x^*, \lambda^*) w > 0, \qquad \forall\ w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0.$$

Problem 1 (4+6+4 Linear Least Square). Consider the inconsistent system of linea equations

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0. \end{aligned}$$

(1) Write down the formulation for the least square solution of this problem, i.e., find A and b in the minimization problem $\min \|Ax - b\|_2$.

(2) Find the equation that the minimizer x^* satisfies and find the minimizer $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$.

(3) Calculate $r = Ax - b$, verify that r is orthogonal to the columns of A.

Problem 2 (8+8 Barrier and Penalty).

(a) Consider the problem

$$\min f(x) = -x_1 x_2 \qquad \text{subject to } \; x_1 + x_2 - 1 \le 0.$$

Find the minimizer $x(\mu)$ of the corresponding logarithmic barrier function and the estimated Lagrange Multiplier $\lambda(\mu)$.

(b) Consider the problem

$$\min f(x) = -x_1 x_2 \qquad \text{subject to } \; x_1 + x_2 - 1 = 0.$$

Find the minimizer $x(\mu)$ of the corresponding quadratic penalty function and the estimated Lagrange Multiplier $\lambda(\mu)$.

Problem 3 (8 Duality). Consider the problem

$$\min_{x \in X} \; f(x) = x_1 \log \frac{x_1}{a_1} + x_2 \log \frac{x_2}{a_2}, \qquad \text{subject to} \quad c_1 x_1 + c_2 x_2 = b$$

where a_1 and a_2 are positive and $X = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$. Find the dual problem, i.e., the objective function $q(\lambda)$ and constraint(s) on λ, assuming no Lagrange Multipliers for the constraint $x \in X$.

Problem 4 (10 Projection) The global minimizer of the convex optimization problem

$$\min_{x \in \Omega} \; f(x) = x_1 + x_2, \qquad \Omega = \{x \in \mathbb{R}^2 \mid \|x\| \le 2\},$$

can be found by the iterative method $x^{k+1} = P_\Omega(x^k - \nabla f(x^k))$, where $P_\Omega(x)$ is the projection of x on the ball Ω. Given $x^0 = (0, 1/2)$, find x^1 and x^2.

Problem 5 (4+4+6 Convexity and Duality)

(a) Is the set $S = \{(x_1, x_2) \mid x_2 \le x_1^3 - x_1\}$ convex?

(b) Is the function $f(x) = |x_1 + x_2| + \min(|x_1|, |x_2|)$ convex?

(c) Consider the following pair of primal and dual linear programming (you should recognize this)

$$
(\mathcal{P}) \quad \begin{array}{ll} \min & 5x_1 + 11x_2 + 8x_3 \\ \text{subject to} & 2x_1 + 4x_2 + 3x_3 \geq 5, \\ & 3x_1 + x_2 + 4x_2 \geq 4, \\ & x_1 + 2x_2 + 2x_3 \geq 3, \\ & x_1, x_2, x_3 \geq 0, \end{array}
\qquad
(\mathcal{D}) \quad \begin{array}{ll} \max & 5y_1 + 4y_2 + 3y_3 \\ \text{subject to} & 2y_1 + 3y_2 + y_3 \leq 5, \\ & 4y_1 + y_2 + 2y_3 \leq 11, \\ & 3y_1 + 4y_2 + 2y_3 \leq 8, \\ & y_1, y_2, y_3 \geq 0. \end{array}
$$

Given that the (global) minimizer for the primal problem $(\mathcal{P})$ is $x^* = (1, 0, 1)$, what's the maximizer y^* for the dual problem $(\mathcal{D})$?

Problem 6 (12 Nonsmooth functions) Let a and b be two positive constants. Find the minimizer of the function

$$f(x, y) = \frac{(x-a)^2 + (y+b)^2}{2} + \max(|x|, |y|).$$

Problem 7 (6+8) Let a_1, a_2, a_3 be positive constants.

(a) Find the maximal value of $x_1 x_2 x_3$ subject to the constraints

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} = 1, \qquad x_1, x_2, x_3 \geq 0.$$

You can assume the constraints $x_1, x_2, x_3 \geq 0$ are inactive at the extremer x^*.

(b) Show that the second order sufficient condition is satisfied at the minimizer x^* you find.

Problem 8 (4+8 Violation of optimality conditions) Consider the function $f(x) = e^{x_1+x_2} + (x_1 - 1)(x_2 - 1)$.

1. Show that $(0, 0)$ is neither a local minimizer nor a local maximizer. In other words, some necessary optimality condition(s) is violated.

2. Therefore should be two directions $\vec{d}_+$ and $\vec{d}_-$ at $(0, 0)$ along which the function f is increasing and decreasing respectively. Find these two directions.