Quadratic programming (Equality constraint)

$$\begin{aligned} \min \quad & q(x) = \frac{1}{2}x^t Qx + c^t x \\ \text{subject to} \quad & Ax = b. \end{aligned}$$

The first order optimality condition

$$\begin{bmatrix} Q & -A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

Is this big matrix positive definite? Does it have a solution?

- Direct solution (Gauss Elimination) of the above system $\Longrightarrow$ Numerical Linear Algebra
- Reduction of variables using $x = \bar{x} + Zv$
- Projected method

Quadratic programming (mixed constraints)

$$
\begin{array}{lll}
\min & q(x) = \frac{1}{2}x^t Qx + c^t x & \\
\text{subject to} & a_i^t x = b_i, & i \in \mathcal{E}, \\
& a_i^t x \geq b_i, & i \in \mathcal{I}.
\end{array}
$$

The Lagrange function

$$L(x, \lambda) = \frac{1}{2}x^t Qx + c^t x - \sum_{i \in \mathcal{I} \bigcup \mathcal{E}}$$

and active set at x^*

$$\mathcal{A}(x^*) = \{i \in \mathcal{E} \bigcup \mathcal{I} \mid a_i^t x^* = b_i\}.$$

The optimality conditions

$$
\begin{aligned}
Qx^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i &= 0, & \\
a_i^t x_i^* &= b_i, & \forall i \in \mathcal{A}(x^*), \\
a_i^t x_i^* &\geq b_i, & \forall i \in \mathcal{I} \backslash \mathcal{A}(x^*), \\
\lambda_i^* &\geq 0, & \forall i \in \mathcal{I} \bigcap \mathcal{A}(x^*),
\end{aligned}
$$

Convex QP: Active-set methods

If we know $\mathcal{A}(x^*)$, then we can solve the equivalent problem

$$\min_x \; q(x) = \frac{1}{2}x^t Qx + c^t x \quad \text{subject to } a_i^t x = b_i, \quad i \in \mathcal{A}(x^*).$$

In general we only have a *working set* $\mathcal{W}_k$ at x_k. We can search in this subset of active constraints, until some of the rest constraints become active. Denote p and define

$$p = x - x_k, \quad g_k = Qx_k + c$$

then $q(x) = q(x_k + p) = \frac{1}{2}p^t Qp + g_k^t p + \rho_k$. Solving the subproblem

$$\begin{aligned} &\min_p && \frac{1}{2}p^t Gp + g_k^t p \\ &\text{subject to} && a_i^t p = 0, \qquad i \in \mathcal{W}_k. \end{aligned}$$

Choose α_k in $x_{k+1} = x_k + \alpha_k p_k$, such that $\mathcal{W}_k$ changes.