Continuous Optimization

Convex optimization problems: Linear and Quadratic programming

Sections covered in the textbook (2nd edition):

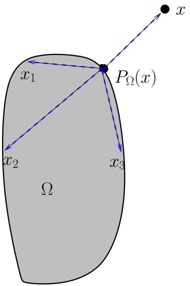
- Chapter 14
- Chapter 15
- Chapter 16

Basic definitions

- Convex sets, convex functions
- Equivalent definitions for smooth convex functions
  (a) $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
  (b) ∇f is monotone: $(\nabla f(x) - \nabla f(y), y - x) \geq 0$
  (c) If f is smooth, $\nabla^2 f(x)$ is nonnegative definite
- Other common convex functions: x^p on $(0, \infty)$ for $p < 0$ or $p \geq 1$; norms $\|x\|_p$ (including $|x|$); e^x, $\cdots$
- Operations on convex functions
  (i) If h and g are convex, then so are $m(x) = \max(f(x), g(x))$ and $h(x) = f(x) + h(x)$
  (ii) If f and g are convex and g is non-decreasing, then $h(x) = g(f(x))$ is convex
  (iii) If $f(x, y)$ is convex in x then $g(x) = \sup_{y \in C} f(x, y)$ is convex

Projection of x on the convex set Ω

Find a point $P_\Omega(x) \in \Omega$ to minimize $\|y - x\|_2$ for any $y \in \Omega$.



Can you guess the sign of $(x_i - P_\Omega(x), x - P_\Omega(x))$?

What is $P_\Omega(x)$ if $x \in \Omega$?

What if Ω is a subspace?

Projection of x on the convex set Ω

Characterization of the projection $P_\Omega(x)$ of x on Ω:

Non-expansive of P_Ω:

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|.$$

Projection on special convex sets:

(a) The unit ball $\|x\|_p \leq 1$, especially $p = 1, 2, \infty$

(b) The positive cone $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) \mid x_i \geq 0\}$

(c) Graph G of a convex function g, for example

$$\{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq t\}$$

Convex optimization problems

Standard form:

$$\min_{x \in \Omega} \quad f(x)$$

where $f(x)$ is a convex function and Ω is a convex set. Or

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \cdots, m, \end{aligned}$$

where f and g_i are convex.

Notice (1) the standard form is alway **min**
(2) the constraints are always "$\leq$" (could be "$=$" for linear constraints)
(3) it is possible to convert nonconvex into convex

Convex optimization problems

Properties

(a) If a local minimal exists, it is a global minimum (but may not be strict)

(b) the set of all (global) minima is convex

(c) For each strictly convex function, if the function has a minimum, then it is unique.

Examples

(i) Linear Programming

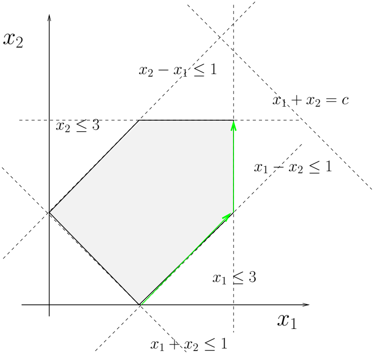
(ii) Linear least squares (with or without linear constraints)

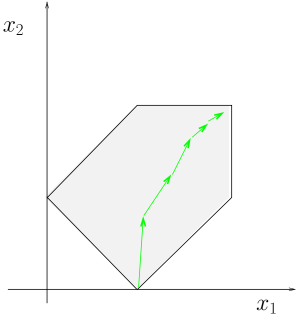
(iii) Convex quadratic minimization with linear constraints

(iv) $\cdots$

Linear Programming

Simplex method vs Interior-point method





$$
\begin{array}{ll}
\text{max} & x_1 + x_2 \\
\text{subject to} & x_1 \le 3,\ x_3 \le 2,\ x_1 + x_2 \ge 1 \\
 & x_1 - x_2 \le 1, \quad x_2 - x_1 \le 1.
\end{array}
$$

Interior-point method

Primal problem:

$$\min \ c^t x, \qquad \text{subject to } Ax = b, x \geq 0.$$

Dual problem:

$$\max b^t \lambda, \ \text{subject to} \quad A^t \lambda + s = c, \ \ s \geq 0.$$

KKT conditions:

$$\begin{aligned} A^t \lambda + s &= c, \\ Ax &= b \\ x_i s_i &= 0, \ \ i = 1, 2, \cdots, n, \\ x, s &\geq 0. \end{aligned} \tag{1a}$$

Interior-point method

Alternative form for the primal-dual form for interior-point method

$$F(x,\lambda,s) = \begin{bmatrix} A^t\lambda + s - c \\ Ax - b \\ XSe \end{bmatrix} = 0, \qquad x, s \geq 0,$$

where $e = (1, 1, \cdots, 1)^t$,

$$X = \mathrm{diag}(x_1, x_2, \cdots, x_n), \qquad S = \mathrm{diag}(s_1, s_2, \cdots, s_n).$$

Basic algorithm: find (x^k, λ^k, s^k) iteratively.

The name interior-point method comes from the fact that $x^k > 0$ and $s^k > 0$. Theoretically you never get exact answer in finite number of iterations, but this prevents certain difficulties and accelerates the convergence for large scale problems

Interior-point method

Recall general algorithms for unconstrained optimization: (1) decreasing direction (2) step length

At $(x^k, \lambda^k, s^k$, define

$$r_b^k = Ax^k - b, \quad r_c^k = A^t\lambda^k + s^k - c$$

then from $0 = F(x^k + \Delta x^k, \lambda^k + \Delta\lambda^k, s^k + \Delta s^k)$, the direction can be computed as

$$\begin{bmatrix} 0 & A^t & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta\lambda^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -r_c^k \\ -r_b^k \\ -X^k S^k e \end{bmatrix}.$$

The step length α^k is chosen such that

$$(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k, \lambda^k, s^k) + \alpha_k(\Delta x^k, \Delta\lambda^k, \Delta s^k).$$

Interior-point method

Given (x^0, λ^0, s^0) with $x^0, s^0 > 0$;
for $k = 0, 1, 2, \cdots$ **do**

Choose $\sigma_k \in [0, 1]$ and solve

$$\begin{bmatrix} 0 & A^t & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -r_c^k \\ -r_b^k \\ -X^k S^k e + \sigma_k \mu_k e \end{bmatrix},$$

where $\mu_k = (x^k)^t s^k / n$; ;
Set

$$(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k, \lambda^k, s^k) + \alpha_k (\Delta x^k, \Delta \lambda^k, \Delta s^k)$$

choosing α_k so that $x^{k+1}, s^{k+1} > 0$.

end

Interior-point method

Other forms:

(a) How about inequality constraints?

$$\min \; c^t x, \qquad \text{subject to } Ax \geq b, x \geq 0.$$

or

$$\min \; c^t x, \qquad \text{subject to } Ax \geq b.$$

(b) How about the penalty form?

$$\min \; c^t x - \tau \sum_{i=1}^{n} \ln x_i, \qquad \text{subject to } Ax = b.$$