Math 309 Assignment 5 Solution

Problem 1. (a) We need one (free) dual variable λ for the equality constraint $x_1 + 2x_2 + 3x_3 = 6$ and three non-negative dual variables $s = (s_1, s_2, s_3)$ for $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. Therefore the original problem is equivalent to

$$\min_{x \text{ free}} \max_{\substack{\lambda \text{ free,} \\ s_1 \geq 0, s_2 \geq 0, s_3 \geq 0}} x_1 + x_2 - \lambda(x_1 + 2x_2 + 3x_3 - 6) - s_1 x_1 - s_2 x_2 - s_3 x_3$$

The dual problem is obtained by interchange the order of min and max, i.e.,

$$\max_{\substack{\lambda \text{ free,} \\ s_1 \geq 0, s_2 \geq 0, s_3 \geq 0}} \min_{x \text{ free}} x_1 + x_2 - \lambda(x_1 + 2x_2 + 3x_3 - 6) - s_1 x_1 - s_2 x_2 - s_3 x_3$$

The coefficients of both x_1, x_2 and x_3 must vanish (or taking the gradient), which gives the condition

$$1 - \lambda - s_1 = 0, \quad 1 - 2\lambda - s_2 = 0, \quad -3\lambda - s_3 = 0.$$

The objective function for the dual problem is simplified as 6λ. Therefore, the dual problem is

$$\begin{array}{ll} \max & 6\lambda \\ \text{subject to} & \lambda + s_1 = 1, \\ & 2\lambda + s_2 = 1, \\ & 3\lambda + s_3 = 0, \\ & s_1 \geq 0, \ s_2 \geq 0, \ s_3 \geq 0. \end{array}$$

(b) The step length α is determined by the the constrants that $x^0 + \alpha\Delta x \geq 0$ and $s^0 + \Delta s \geq 0$ (λ is free, no conditions for it). This implies the following inequalities

$$x^0 + \alpha\Delta x = \begin{pmatrix} 4+\alpha \\ 1+\alpha \\ -\alpha \end{pmatrix} \geq 0, \quad s^0 + \alpha\Delta s = \begin{pmatrix} 2-\alpha \\ 3-2\alpha \\ 3-3\alpha \end{pmatrix} \geq 0.$$

The intersection of all these inequalities is $\alpha = 0$ (α should be non-negative) and we have $x^1 = x^0$.

(c) Since the complementarity conditions $x_i s_i = 0$ are obviously violated at x^1, this is not a local minimizer.

2. Since β_μ is unconstrained, the local minimizer is given by

$$\beta_\mu'(x) = \frac{1}{x+1} - \frac{\mu}{x} = 0$$

or the minimizer (depending on μ) is $x(\mu) = \mu/(1-\mu)$. We have

$$\lim_{\mu \to 0^+} x(\mu) = 0 = x^*,$$

which is the global minimizer.

3. (a) If we choose c_1 as the only active constraint, then the subproblem is

$$\begin{array}{ll} \max & \frac{1}{2}(x_1-3)^2 + (x_2-2)^2 \\ \text{subject to} & 2x_1 - x_2 = 0. \end{array}$$

The minizer is $x^1 = (11/9, 22/9)$ and the Lagrange Multiplier is $\lambda_1 = -8/9 < 0$. This implies that c_1 can not be a local minimizer of the original problem and c_1 is not active.

(b) We have an unconstrained problem at x^1 with

$$\nabla f(x^1) = \begin{pmatrix} -16/9 \\ 8/9 \end{pmatrix}, \qquad \nabla^2 f(x^1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The search direction is given by $p^1 = -(\nabla^2 f(x^1))^{-1} \nabla f(x^1) = \begin{pmatrix} 16/9 \\ -4/9 \end{pmatrix}$. The step length α is determined either when the global minimizer is obtained inside the feasible region or $x^1 + \alpha p^1$ stays on the boundary of the feasible set (some constraints become active). For this case, the global minimizer along the line $x^1 + \alpha p^1 = (\frac{11-16\alpha}{9}, \frac{22+8\alpha}{9})$ is outside the feasible region. we have to choose the largest non-negative α in $x^1 + \alpha p^2$ while satisfying the constraints,

$$2\frac{11-16\alpha}{9} - \frac{22+8\alpha}{9} \geq 0, \quad -\frac{11-16\alpha}{9} - \frac{22+8\alpha}{9} \geq -4, \quad -\frac{22+8\alpha}{9} \geq 0,$$

or $0 \leq \alpha \leq 1/4$. Therefore $\alpha^1 = 1/4$, $x^2 = (5/3, 7/3)$ and c_2 becomes active. Finally if we take c_2 as the only active constraint, we can get the global minimizer $x^* = (7/3, 5/3)$.