MATH 309 Assignment 6

No need to hand in, the solution will be released on April 10th.

1. Consider the interior point method for the linear programming

$$\begin{array}{ll} \min & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 = 6, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array}$$

   (a) What's the dual problem, in terms of λ and $s = (s_1, s_2, s_3)$? Write down all the objective function and constraints using numbers (not the "abstract" formulation with A, b and c).

   (b) If we start the interior point method with

$$x^0 = (4, 1, 0),\ \lambda^0 = -1,\ s^0 = (2, 3, 3)$$

   and the increment

$$\Delta x = (1, 1, -1),\ \Delta\lambda = 1,\ \Delta s^0 = (-1, -2, -3).$$

   How large can you choose α so that $(x^1, \lambda^1, s^1) = (x^0, \lambda^0, s^0) + \alpha(\Delta x, \Delta\lambda, \Delta s)$ is still inside the feasible region (for both primal and dual problem)?

   (c) Do you get the optimal solution (x^*, λ^*, s^*) by choosing the largest possible α calculated in (b)?

2. Consider the one-dimensional optimization problem

$$\begin{array}{ll} \min & f(x) = \log(x + 1) \\ \text{subject to} & x \geq 0. \end{array}$$

   Find the minimizer x^*_μ with the logarithmic barrier

$$\beta_m u(x) = \log(x + 1) - \mu \log(x),$$

   for $0 < \mu < 1$. What's the limit of x^*_μ when $\mu \to 0$?

3. (Active-set method.) Consider the problem

$$\begin{array}{lll} \min & f(x) = \frac{1}{2}(x_1 - 3)^2 + (x_2 - 2)^2 & \\ \text{subject to} & 2x_1 - x_2 \geq 0, & (c_1) \\ & -x_1 - x_2 \geq -4, & (c_2) \\ & x_2 \geq 0. & (c_3) \end{array}$$

In the lecture, starting with $x^0 = (0,0)$ by assuming both c_1 and c_3 are active, we get rid of c_1 and then find the sequence of points $x^1 = (3,0)$, $x^2 = (3,1)$ and finally the global minimizer $x^* = (7/3, 5/3)$. This exercise ask you to do the same problem by get rid of c_3 during the first step.

(a) If c_1 is the only active set, what's the minimizer x^1? Show that this is not a local minimizer of the original problem (the Lagrangian multiplier has the wrong sign).

(b) The previous step implies that c_1 is not active and we treat this as a unconstrained problem using Newton's method. Find the decreasing direction $p = -(\nabla^2 f(x^1))^{-1}\nabla f(x^1)$ from Newton's method and choose the largest α such that $x^1 + \alpha p$ is still inside the feasible region. Now c_2 becomes active now.

You don't have to do the last step (consider the subproblem that only c_2 is active), because we get the same global minimizer $x^* = (7/3, 5/3)$ as in the lecture.