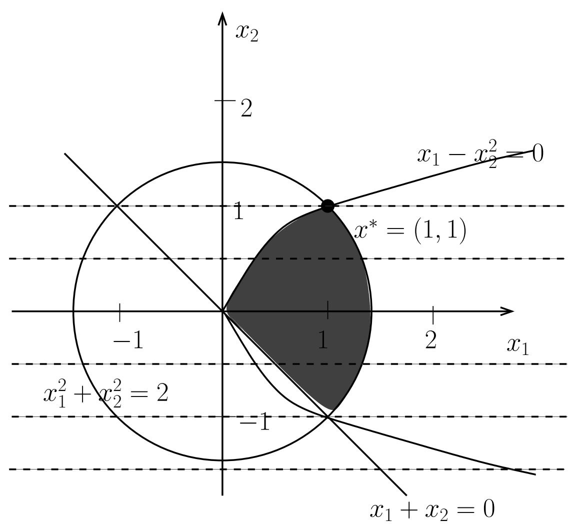
Math 309 Assignment 5 Solution

Problem 1. (a) The global minimizer is $x^* = (-1, -1)$.



(b) At x^*, the first two constraints are active and the third one is inactive. Then $\lambda_3^* = 0$ and from the Lagrange function

$$L(x_1, x_2, \lambda_1, \lambda_2) = -x_2 - \lambda_1(2 - x_1^2 - x_2^2) - \lambda_2(x_1 - x_2^2),$$

we have

$$\nabla_x L(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*) = \begin{pmatrix} 2\lambda_1^* x_1^* - \lambda_2^* \\ -1 + 2\lambda_1^* x_2^* + 2\lambda_2^* x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $x^* = (1, 1)$ into above equation, we have $\lambda_1^* = 1/6, \lambda_2^* = 1/3$. It is easy to see that x^* is feasible, $\lambda^* \geq 0$, the strict complementarity and LICQ conditions are satisfied.

Since at x^*, $\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are two linear indepedent vectors in $\mathbb{R}^2$. This implies that the critical cone $\mathcal{C}(x^*, \lambda^*) = \{w \mid w^t \nabla c_1(x_1^*) = 0, w^t \nabla c_2(x_1^*) = 0\} = \{0\}$. Therefore, the second order sufficient condition is satisfied too. Therefore x^* is a local minimizer.

Problem 2. (a) Since $\nabla c_1(x^*) = (0, 1)^t, \nabla c_2(x^*) = (0, 1)^t = \nabla c_1(x^*)$, the linearized feasible direction $\mathcal{F}(x^*)$ is

$$\{w \mid w^t \nabla c_1(x^*) = w^t \nabla c_2(x^*) = 0\} = \{(d, 0)^t \mid d \in \mathbb{R}\}.$$

Since the feasible region Ω contains only one point $(0,0)^t$, any feasible sequence $\{z_n\}$ is exactly $z_n = (0,0)^t$ and the tangent cone $\mathcal{T}_\Omega(x^*) = \{(0,0)^t\}$, which is different frome $\mathcal{F}(x^*)$.

(b) The Lagrange multipliers satisfy the equation

$$\nabla L(x^*, \lambda^*) = \begin{pmatrix} 3\lambda_2^* x_1^{*2} \\ 1 - \lambda_1^* - \lambda_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, there are infinity many solutions, parameterized by $(\lambda_1^*, \lambda_2^*) = (t, 1-t), t \in \mathbb{R}$.

Problem 3. (a) The original problem is equivalent to

$$\min_{x\in X} \max_{\lambda} \ L(x,\lambda) = \min_{x\in X} \max_{\lambda} \ x_1 \log\frac{x_1}{a_1} + x_2 \log\frac{x_2}{a_2} - \lambda(c_1x_1 + c_2x_2 - b)$$

and the dual function is defined as

$$q(\lambda) = \min_{x\in X} L(x,\lambda) = \min_{x\in X} x_1 \log\frac{x_1}{a_1} + x_2 \log\frac{x_2}{a_2} - \lambda(c_1x_1 + c_2x_2 - b).$$

For fixed λ, the minimizer satisfies

$$\nabla_x L(x,\lambda) = \begin{pmatrix} \log\frac{x_1}{a_1} + 1 - \lambda c_1 \\ \log\frac{x_2}{a_2} + 1 - \lambda c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$x_1 = a_1 e^{\lambda c_1 - 1}, \quad x_2 = a_2 e^{\lambda c_2 - 1}.$$

Substituting it into L, we get

$$\begin{aligned} q(\lambda) &= a_1(\lambda c_1 - 1)e^{\lambda c_1 - 1} + a_2(\lambda c_2 - 1)e^{\lambda c_2 - 1} - \lambda(a_1c_1e^{\lambda c_1 - 1} + a_2c_2e^{\lambda c_2 - 1} - b) \\ &= \lambda b - a_1 e^{\lambda c_1 - 1} - a_2 e^{\lambda c_2 - 1}. \end{aligned}$$

The first and second order derivatives of q are

$$q'(\lambda) = b - a_1c_1e^{\lambda c_1 - 1} - a_2c_2e^{\lambda c_2 - 1}, \quad q''(\lambda) = -a_1c_1^2e^{\lambda c_1 - 1} - a_2c_2^2e^{\lambda c_2 - 1}.$$

Problem 4. (a) If (x_1, x_2) and (y_1, y_2) are in Ω, then $x_1^2 \le x_2, y_1^2 \le y_2$. For any $\lambda \in [0,1]$,

$$\lambda x_1^2 + (1-\lambda)y_1^2 - (\lambda x_1 + (1-\lambda)y_1)^2 = \lambda(1-\lambda)(x_1 - y_1)^2 \ge 0. \tag{1}$$

Therefor,

$$(\lambda x_1 + (1-\lambda)y_1)^2 \le \lambda x_1^2 + (1-\lambda)y_1^2 \le \lambda x_2 + (1-\lambda)y_2.$$

This implies that $(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \in \Omega$ and Ω is convex.

(b) Since $p \notin \Omega$, the projection $P_\Omega(p) = (x_1, x_2)$ is on the boundary of Ω, such that $x_2 = x_1^2$. To find this point, we only have to find x_1 such that it minimizes

$$\|p - x\|_2^2 = (x_1 - 3)^2 + (x_1^2 - 0)^2 = x_1^4 + x_1^2 - 6x_1 + 9 \stackrel{\text{def}}{=} f(x_1),$$

which is given by the roots of $f'(x_1) = 4x_1^3 + 2x_1 - 6 = 0$. The only root is given by $x_1 = 1$ and therefore the projection of $(3,0)$ on Ω is $(1,1)$.