MATH 309 Assignment 5

Due March 30

1. Consider the problem

$$\begin{aligned} \min \quad & f(x) = -x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 2, \\ & -x_1 + x_2^2 \leq 0, \\ & x_1 + x_2 \geq 0. \end{aligned}$$

   (a) Plot the feasible region, contours of the objective function and find the global minimizer graphically.

   (b) Show that the minimizer satisfies all the first and second order optimality conditions.

2. Consider the problem

$$\begin{aligned} \min \quad & f(x) = x_2 \\ \text{subject to} \quad & x_2 = 0 \\ & x_2 - x_1^3 = 0. \end{aligned}$$

   The global minimizer is obviously $x^* = (0, 0)$ because the feasible region contains only this point.

   (a) Find the tangent cone $\mathcal{T}_\Omega(x^*)$ and linearized feasible region $\mathcal{F}(x^*)$.

   (b) Find $\lambda^* = (\lambda_1^*, \lambda_2^*)$ such that $\nabla L(x^*, \lambda^*) = 0$, where

$$L(x, \lambda) = x_2 - \lambda_1 x_2 - \lambda_2(x_2 - x_1^3).$$

   The problem shows that if $\mathcal{T}_\Omega(x^*)$ is different from $\mathcal{F}(x^*)$, the Lagrange Multiplier λ^* could exist such that $\nabla L(x^*, \lambda^*) = 0$.

3. Consider the problem

$$\min_{x \in X} f(x) = x_1 \log \frac{x_1}{a_1} + x_2 \log \frac{x_2}{a_2}, \qquad \text{subject to} \quad c_1 x_1 + c_2 x_2 = b$$

   where a_1 and a_2 are positive and $X = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$.

   (a) Find $q(\lambda)$ in the dual problem, assuming no Lagrange Multipliers for the constraint $x \in X$.

   (b) Find the first and second order derivatives of q.

4. (a) Show that the set

$$\Omega = \{(x_1, x_2) \mid x_1^2 \leq x_2\}$$

is convex.

(b) Find the projection of the point $p = (3, 0)$ on Ω (You have to solve a cubic equation. You can either using the formula for the roots of cubic equation, or just verify the root. No need to show it is unique).