Math 309 Assignment 4 Solution

Problem 1. (a) The Lagrange function is

$$L(x,\lambda) = \frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + \frac{1}{2}x_3^2 - x_3 - \lambda(2x_1 + 3x_2 + x_3 - 7).$$

Therefore the minimizer satisfies

$$\nabla_x L(x,\lambda) = \begin{pmatrix} x_1 + x_2 - 2\lambda \\ 2x_2 + x_1 - 3\lambda \\ x_3 - 1 - \lambda \end{pmatrix} = 0$$

The solution x^* can be written in terms of λ^*, i.e.,

$$x_1^* = \lambda^*,\ x_2^* = \lambda^*,\ x_3^* = 1 + \lambda^*.$$

Substitute it into the constraint,

$$7 = 2x_1^* + 3x_2^* + x_3^* = 6\lambda^* + 1$$

Therefore $\lambda^* = 1$ and the minimizer is $x^* = (1,1,2)$. The minimal value is $f(x^*) = 5/2$.

(b) (There is a typo in the problem, the new constraint only have a perturbation on the right hand side, from 7 to $7+\delta$). Let the new minimizer and Lagrange multiplier under the new constraint be $2x_1 + 3x_2 + x_3 = 7 + \delta$ be x_δ^* and λ_δ^*, respectively. Then from the relation

$$\lambda^* = \frac{d}{d\delta} L(x_\delta^*, \lambda_\delta^*)\Big|_{\delta=0} = 1,$$

we have

$$L(x_\delta^*, \lambda_\delta^*) = L(x^*, \lambda^*) + \lambda^*\delta + O(\delta^2) = \frac{5}{2} + \delta + O(\delta).$$

Remark. You can solve the unperturb problem and should get the same answer.

Problem 2. (a) The (linearly indepedent) columns Z are in null space of the linear constraint, i.e., $2x_1 + 3x_2 + x_3 = 0$. Since this is one equation with three unknowns, ther are two parameters in the general solution, or Z has two columsn, and is given by

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix}$$

and $v = (v_1, v_2)$. When $\phi(v) = f(x^* + Zv)$,

$$\nabla\phi(0) = Z^t\nabla f(x^*) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b)

$$\begin{aligned} Z^t\nabla^2 f(x^*)Z &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 7 \\ 7 & 12 \end{pmatrix}. \end{aligned}$$

Since the trace and the determinant are both positive, $Z^t\nabla^2 f(x^*)Z$ has two positive eigenvalues, and it is positive definite.

Problem 3. (a) If the only constraint is c_1, we can solve the following equivalent problem

$$\begin{array}{ll} \text{minimizer} & f(x) = \frac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} & -x_1 + x_2 = -1. \end{array}$$

Define the Lagrange function

$$L(x, \lambda_1) = \frac{1}{2}x_1^2 + x_2^2 - \lambda_1(-x_1 + x_2 + 1).$$

and the minimizer satisfies

$$\nabla_x L(x, \lambda_1) = \begin{pmatrix} x_1 + \lambda_1 \\ 2x_2 - \lambda_1 \end{pmatrix} = 0$$

Therefore, $\hat{x}_1^* = -\lambda_1$ and $\hat{x}_2^* = \lambda_1/2$. Substitute it into the constraint,

$$-1 = -\hat{x}_1^* + \hat{x}_2^* = \frac{3}{2}\lambda_1,$$

or $\hat{\lambda}_1^* = -2/3$ and $\hat{x}^* = (2/3, -1/3)$. Since $\hat{x}^*$ does not satisfy the second constraint (it is not in the feasible region), it does not satisfy the necessary conditions.

(b) When only the second constraint is active, then the problem is equivalent to

$$\begin{array}{ll} \text{minimizer} & f(x) = \frac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} & 2x_1 + x_2 = 2. \end{array}$$

The Lagrange function is

$$L(x, \lambda) = \frac{1}{2}x_1^2 + x_2^2 - \lambda_2(2x_1 + x_2 - 2)$$

The minimizer is given by

$$\nabla_x L(x, \lambda_2) = \begin{pmatrix} x_1 - 2\lambda_2 \\ 2x_2 - \lambda_2 \end{pmatrix} = 0$$

or $\tilde{x}_1^* = 2\tilde{\lambda}_2^*, \tilde{x}_2^* = \tilde{\lambda}_2^*/2$.

Substitute it into the constraint

$$2 = 2\tilde{x}_1^* + \tilde{x}_2^* = \frac{9}{2}\tilde{\lambda}_2^*,$$

Therefore $\tilde{\lambda}_1^* = 4/9$ and $\tilde{x}^* = (8/9, 2/9)$. Since $\tilde{x}^*$ satisfies the first constraint, it is inside the feasible region. Moreover, the first constraint is a stric inequality at $\tilde{x}^*$, from complementarity condition $\tilde{\lambda}_1^* = 0$. The necessary conditions $\tilde{\lambda}_i^* \geq 0$ and $\tilde{\lambda}_i c_i(\tilde{x}^*) = 0$ are satisfied.

Next we have

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} 2 & 1 \end{pmatrix}.$$

Thefore

$$\nabla f(\tilde{x}^*) = \begin{pmatrix} 8/9 \\ 4/9 \end{pmatrix} = \hat{A}^t \tilde{\lambda}_2^*.$$

Finally the matrix Z is obtained from the null space of $\hat{A}$, i.e. $Z = (1, -2)^t$ and

$$\hat{Z}^t \nabla^2 f(\tilde{x}^*) \hat{Z} = (1, -2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 9 \geq 0.$$

Therefore, $\tilde{x}^*$ satisfies all the necessary conditions.

Problem 4. There are two extra conditions to check.

(1) Strict complementarity: since for $i = 1, 2$, only one of $\tilde{\lambda}_i^*$ and $c_i(\tilde{x}^*)$ is zero, this condition is satisfied.

(2) Positive definite of $Z^t \nabla^2 f(\tilde{x}^*) Z$. This is obvious from the calculation in the previous problem.

Putting them altogether, $\tilde{x}^*$ is a local minimizer.