Math 309 Assignment 4

Due on March 13

Problem 1(4+2 pt). (a) Use Lagrange Multiplier to find the minimizer x^* of the problem

$$\begin{array}{rc} \text{minize} & f(x) = \frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + \frac{1}{2}x_3^2 - x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 = 7 \end{array}$$

and find $f(x^*)$.

(b) If the constraint is changed to $2x_1 + 3x_2 + x_3 = 7 + \delta$ for some small number δ (i.e. $|\delta| \ll 1$) and the minimizer becomes x_δ^*. Find an approximate of $f(x_\delta^*)$ up to the linear order of δ. (We know that $f(x_\delta^*)$ depends on δ on the following way $f(x_\delta^*) = f_0 + f_1\delta + f_2\delta^2 + \cdots$. Find f_0 and f_1.)

Problem 2(2+2 pt). Consider the same problem as in Problem 1

$$\begin{array}{rc} \text{minize} & f(x) = \frac{1}{2}x_1^2 + x_2^2 + x_1x_2 + \frac{1}{2}x_3^2 - x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 = 7. \end{array}$$

(a) The general solution can be wrtten as $x = x^* + Zv$, where x^* is the minimizer found above. What's the matrix Z. Definte $\varphi(v) = f(x^* + Zv)$, show that

$$\nabla\varphi(0) = Z^t\nabla f(x^*) = 0.$$

(b) Show that the matrix $Z^t\nabla^2 f(x^*)Z$ is positive definite.

Problem 3 (3+3 pt). Consider the probem

$$\begin{array}{rll} \text{minimize} & f(x) = \frac{1}{2}x_1^2 + x_2^2 & \\ \text{subject to} & -x_1 + x_2 \geq -1 & (c_1) \\ & 2x_1 + x_2 \geq 2 & (c_2) \end{array}$$

(a) Assuming that the the only *active* constraint is c_1. Find the minimizer $\hat{x}^*$ of $f(x)$ under this single constraint c_1. Does $\hat{x}^*$ satisfy all the necessary conditions for linearly constrained problems?

(b) Now Assuming that the the only *active* constraint is c_2. Find the minimizer $\tilde{x}^*$ of $f(x)$ under this single constraint c_2. Does $\tilde{x}^*$ satisfy all the necessary conditions for linearly constrained problems?

You can stop your calculation once you find any of those necessary conditions is violated (it can not be a minizer). You should find one and one minimizer satisfies all the necessary conditions and in general you have to try all the four (instead of two here) cases of different combinations.

Problem 4 (4 pt). Consider the same problem as above

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} \quad & -x_1 + x_2 \geq -1 && (c_1) \\ & 2x_1 + x_2 \geq 2 && (c_2) \end{aligned}$$

Now check whether the minizer calculated in problem 3 satisfies the sufficient conditions. You only have to check the additional conditions for sufficiency.