

# Math 309 Assignment 3

Problem 1. (1)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(2) The objective function  $\|Ax - b\|^2$  can be written as

$$f(x) = \|Ax - b\|^2 = (Ax - b)^t(Ax - b) = (x^t A^t - b^t)(Ax - b) = x^t A^t A x - 2b^t A x + b^t b$$

and therefore the minimizer satisfies

$$\nabla f(x^*) = A^t A x^* - A^t b = 0,$$

where

$$A^t A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A^t b = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Therefore, the minimizer  $x^* = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$ .

(3)

$$r = Ax - b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \\ 0 \end{pmatrix}.$$

The two columns of  $A$  are  $c_1 = (1, 0, 1, 1)^t$  and  $c_2 = (0, 1, 1, -1)^t$ . It is obvious that  $c_1^t r = 0$  and  $c_2^t r = 0$ .

Problem 2. The Lagrange function is

$$L(a, b, \lambda_1, \lambda_2) = a^t b - \lambda_1(\|a\|_2 - A) - \lambda_2(\|b\|_2 - B).$$

Taking the gradient with respect to  $a$  and  $b$ , we have

$$\nabla_a L = b - \lambda_1 \frac{a}{\|a\|_2} = 0, \quad \nabla_b L = a - \lambda_2 \frac{b}{\|b\|_2} = 0$$

Therefore,

$$b = \lambda_1 \frac{a}{\|a\|_2} = \frac{\lambda_1}{A} a$$

and  $\|b\|_2 = \left\| \frac{\lambda_1}{A} a \right\|_2 = \frac{|\lambda_1|}{A} \|a\|_2 = |\lambda_1|$ . Since the components of  $a$  and  $b$  are positive,  $\lambda_1 > 0$

and hence  $\lambda_1 = B$ . Using the optimal relation,  $b = \frac{\lambda_1}{A} a = \frac{B}{A} a$ , we have

$$\max a^t b = \frac{B}{A} a^t a = AB.$$

Problem 3. The Lagrange function is

$$L(x_1, x_2, \lambda) = (a + 2)x_1 + 4x_2 - \lambda [a(x_1 + e^{x_1}) + b(x_2 + e^{x_2}) - 1].$$

Taking the partial derivatives of  $L$  w.r.t  $x_1$  and  $x_2$  then

$$\begin{cases} \frac{\partial L}{\partial x_1} &= a + 2 - \lambda a - \lambda a e^{x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= 4 - \lambda b - \lambda b e^{x_2} = 0 \end{cases}$$

Since  $(0, 0)^t$  is the minimizer, we have

$$\begin{cases} 0 &= \frac{\partial L}{\partial x_1}(0, 0, \lambda) = a + 2 - 2\lambda a \\ 0 &= \frac{\partial L}{\partial x_2}(0, 0, \lambda) = 4 - 2\lambda b \end{cases} \quad (1)$$

together with the constraint  $a + b = 1$ .

The first equation in (1) can be written as

$$\lambda(a + 1) = a + 2 \quad (2)$$

and the second equation in (1) can be reduced to

$$\frac{a + 2}{4} = \frac{2\lambda a}{2\lambda b} = \frac{a}{b} \quad (3)$$

Substute  $b = 1 - a$  into the previous equation, we get a quadratic equation for  $a$ ,

$$a^2 + 5a - 2 = 0$$

or  $a = (-5 \pm \sqrt{33})/2$ .

When  $a = (-5 - \sqrt{33})/2$ ,  $b = 1 - a = (7 + \sqrt{33})/2$ ,  $\lambda = 2/b = (7 - \sqrt{33})/4$  and when  $a = (-5 + \sqrt{33})/2$ ,  $b = 1 - a = (7 - \sqrt{33})/2$ ,  $\lambda = 2/b = (7 + \sqrt{33})/4$ .

**Theorem 0.1.** *Sorry I made a mistake. The calculation with the original objective function  $f(x_1, x_2) = (a + 2)x_1 - 2x_2$  is actually simpler.*

Problem 4. The Lagrange function is

$$L(x, \lambda) = \frac{1}{2} \|x - r\|_2^2 - \lambda(a^t x - b)$$

Therefore, the solution  $x^*$  is given by

$$\nabla_x L(x^*, \lambda) = x^* - r - \lambda a = 0$$

Substitute  $x^* = r + \lambda a$  into the constraint, we have

$$b = a^t(r + \lambda a) = a^t r + \lambda a^t a.$$

This gives  $\lambda = (b - a^t r)/a^t a$  and  $x^* = r + \frac{b - a^t r}{a^t a} a$ .